GEOMETRIC METHODS IN QUANTUM MECHANICS

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Lecture XI - The Sato-Segal-Wilson grassmannian

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THE KLEIN QUADRIC

$Gr(4,2)$ planes $(\mathbb{C}^2)$ in $\mathbb{C}^4$

$\equiv$ lines $(\mathbb{P}^1)$ in $\mathbb{P}^3$

p_1 : (x_0, x_1, x_2, x_3)

p_2 : (y_0, y_1, y_2, y_3)

$$P_{ij} := \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

Plücker coordinates
homogeneous coordinates in $\mathbb{P}^5$

$$\begin{vmatrix} x_0 \dots x_3 \\ y_0 \dots y_3 \\ x_0 \dots x_3 \\ y_0 \dots y_3 \end{vmatrix} = 0 \implies$$

$$P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12} = 0$$

$$\implies \quad Gr(4,2) \cong Q \hookrightarrow \mathbb{P}^5$$

★ Q : Klein quadric

The embedding

$$Gr(4,2) \cong Q \xhookrightarrow[\text{Plücker}]{} \mathbb{P}^5$$

is realized à la Kodaira via Det^*
(dual of det)

$$Gr(4,2) \ni \overline{W} \longmapsto \underbrace{\mathbb{C}\, w_1 \wedge w_2}_{\text{orthonormal basis}} \equiv \underbrace{Det_{\overline{W}}}_{\text{determinant line}}$$

$$\begin{array}{ccc} Det = Pl^* \mathcal{O}(-1) & & \mathcal{O}(-1) \\ \downarrow & & \downarrow \\ Gr(4,2) & \xhookrightarrow[Pl]{} & \mathbb{P}^5 \end{array}$$

tautological bundle

$[v] \mapsto \mathbb{C} v$

$v \neq 0$

Also:

$$Gr(4,2) \cong U(4) / U(2) \times U(2)$$

one realizes an irreducible unitary representation of $U(4)$ on the space of holomorphic sections of Det^*

(Borel - Weil)

- **Infinite dimensional spinors & $\mathcal{I}_{res}$**

(H, g) real Hilbert space ⇝ $Cl_{\mathbb{R}}(H, g)$
real clifford algebra ⇝ $Cl_{\mathbb{C}}(H, g)$
complex clifford algebra

$$\underset{\text{complex Clifford algebra}}{Cl_{\mathbb{C}}(H, g)} \cong \underset{\text{CAR algebra}}{CAR(W)}$$

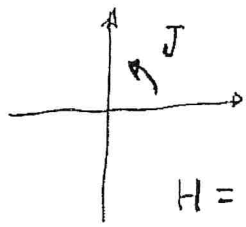
$W < H^{\mathbb{C}}$ isotropic w.r.t. $g^{\mathbb{C}}$

$\wr$

J_W **complex structure** on H

$$(J_W^2 = -I)$$

S : spinors acted on by $CAR(W)$



$\mathbb{R}^2$

$H = \mathbb{R}^2$
$H^{\mathbb{C}} = \mathbb{C}^2$

Majorana - Fock space

ΛW
exterior algebra

infinite dimensional orthogonal group representation theory Brauer Weyl, Cartan, Shale - Stinespring

$$\mathcal{I}_{res}(H^{\mathbb{C}}, W) := \left\{ \begin{matrix} W' \\ \xi \\ J_{W'} \end{matrix} \;\middle/\; P_{W'} - P_W \in L^2 \right\}$$

Shale-Stinespring

restricted isotropic Grassmannian

basic example

$H^{\mathbb{C}} = L^2(S^1, \mathbb{C})$

$W = H_+$

non negativ Fourier modes

O_{res} - homogeneous Kähler manifold

$$S \cong \Gamma^{L^2}_{hol}\left(Pf^* \longrightarrow \mathcal{I}_{res} \right)$$

"bosonization"

Borel - Weil for O_{res}

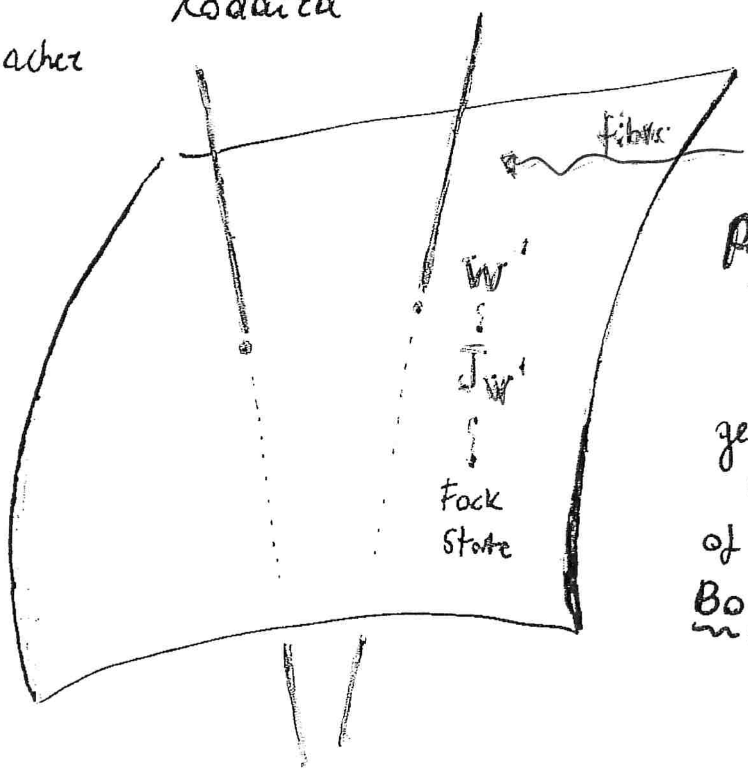
$\tilde{O}_{res}$ univ. central extension

$$\mathcal{I}_{res} \xhookrightarrow{\;i\;} \mathbb{P}(S)$$

Kodaira

— & T. Wurzbacher 1998

S



Pf: Pfaffian line bundle

geometric interpretation of Bogolubov automorphism

• $Gr_{res}(H, H_+)$

$$H = \underset{\infty}{H_+} \oplus \underset{\infty}{H_-}$$

$H = L^2(S^1, \mathbb{C})$

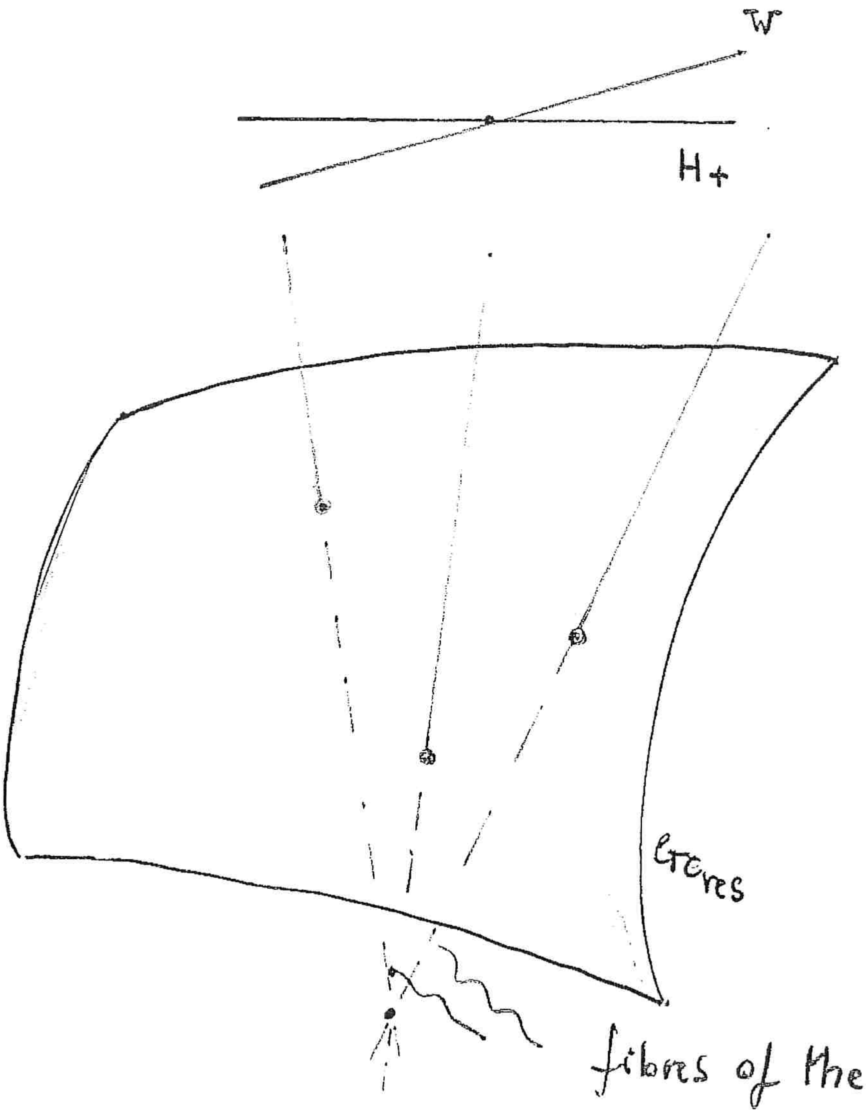
H_+ : non negative fourier modes

$W \in Gr_{res} \iff P_W - P_+ \in HS$

projections

" W "close" to H_+ "

— (cf. Valli '94)



$\mathcal{H}_{P_+}$

|| CAR-algebra representation space pertaining to a (pure) quasi-free state"

fibres of the determinant line bundle Det

" $W \sim w_1 \wedge w_2 \wedge \dots$ "

orthonormal basis

$\mathrm{CAR}(H)$: $[a^*(f), a(g)] = \langle f | g \rangle_H I$

(complex Clifford algebra) $[a(f), a(g)] = 0$

a^* creation operators
a annihilation operators

(fermionic systems)

$$\begin{array}{ccc} \mathrm{Det}^* & & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ Gr_{res} & \longrightarrow & \mathbb{P}(\mathcal{H}_+) \end{array}$$

$\mathcal{O}(1)$ ← hyperplane bundle

Plücker Embedding — cf. Klein quadric

$\lambda \in \Lambda$ $\quad a^*(\lambda)\Lambda = 0 \equiv$ Pauli Exclusion Principle

(–, Valli (1994))

$$\boxed{Gr_{res}(H, H_+) = U_{res}(H) / U(H_+) \times U(H_-)}$$

homogeneous Kähler manifold

$U_{res}(H) = \{ u \in U(H) \mid [u, J] \in HS \}$ $\quad J: P_+ - P_-$

restricted unitary group

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

polarization operator

[Gr_{res} : hermitian symmetric space]

(— & T. Wurzbacher 2000)

SCHEME

FUNCTIONAL ANALYSIS

GEOMETRY

Tools I

* GNS theorem
* CAR algebra
* gauge invariant quasi free states
* The group $U(K)$
* Powers - Størmer's theorem

Dirac

* $Gr(K, K_+)$
* Det

($U_{res}(K)$ - homogeneous)

* Borel Weil :

$$\Gamma^2_\sigma(\mathrm{Det}^*) = H^{A(K)}_{E_+} \quad \text{GNS}$$

* $\tilde{U}_{res}(K)$: central extension of $U_{res}(K)$ by $U(1) \cong S^1$

Majorana

Tools II

* Clifford algebra
* The group $O(H)$
* Bogoliubov transformations
* Shale - Stinespring's theorem

* $\mathcal{J}_{res}(H)$

Siegel manifold

isotropic grassmannian

* Pf

($O_{res}(H)$ hom)

* $\tilde{O}_{res}(H)$ central extension of $O_{res}(H)$ by $U(1)$

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$$\mathcal{S} = \Gamma^2_\sigma(\mathrm{Pf}^*) = H^{A(W)}_{I_W}$$ GNS

Spin^c representation (Borel-Weil)
of $\widetilde{O_{res}}(H)$

$\boxed{\text{* Fock-anti-Fock correspondence}} \longrightarrow \boxed{\mathrm{Det}\Big|_{Y_{res}(H)} = \mathrm{Pf}^{\otimes 2}}$

(co-dim Hodge operator)

* Plücker's Equations
embedding

$$\mathrm{Gr}(K, K_+) \hookrightarrow \mathbb{P}(H)$$
$$\mathbb{P}(H) = H^{A(K)}_{E_+}$$

— —

TOOLS I

- GNS construction

A (unital) C^* algebra — Banach $*$-algebra with unit fulfilling $\|a^*a\| = \|a\|^2$ $\forall a \in A$

ω state on A — positive linear functional of norm 1, $\omega(I) = 1$

$\exists !$ (up to unitary equivalence)

$$\underbrace{\pi_\omega}_{\text{representation}}, \underbrace{H_\omega}_{\text{Hilbert space}}, \underbrace{\xi_\omega}_{\text{cyclic vector}}$$

$(\pi_\omega, H_\omega, \xi_\omega)$ GNS triple

such that

$$\omega(a) = \langle \xi_\omega, \pi_\omega(a)\xi_\omega \rangle_{H_\omega}$$

if ω is faithful ($\omega(a^*a) = 0$ iff $a = 0$)
(≥ 0)

then $H_\omega = \bar{A}$; $\langle a, b \rangle = \omega(a^*b)$, $\xi_\omega = I$

- The CAR algebra   canonical anti commutation relations

K complex separable Hilbert space
one particle space

$f \underset{\text{linear}}{\longmapsto} a(f)^*$ creation operator

$f \underset{\text{antilinear}}{\longmapsto} a(f)$ annihilation operator

$$\begin{cases} [a(f)^*, a(g)]_+ = \langle f | g \rangle I \\ [a(f), a(g)]_+ = 0 \end{cases} \quad \forall f, g \in K$$

- gauge-invariant quasi-free states

$$\omega_E \quad , \quad E = E^2 = E^* \in B(K)$$

completely determined by its 2-pt functions

$$\omega_E(a(f)^* a(g)) = \langle f, Eg \rangle$$

$E = 0$ Fock state (vacuum)

$$H_0 \equiv H_{W_0} = \Lambda K = \bigoplus_{n=0}^{\infty} \Lambda^n K \qquad \Lambda^0 K = \mathbb{C}\,\Omega$$

Fock

$a(f)^* \rightsquigarrow f \wedge$ wedge

$a(f) \rightsquigarrow f \lrcorner$ contraction

$E = I$ anti-Fock state (full)

- Fock – anti-Fock correspondence

$$A(K) \xrightarrow{\cong} A(K^*) \qquad K^*: \text{dual of } K$$

$$a(f) \to a(f^*)^*$$

$$\pi_0 \to \pi_I^*$$

$$\pi_I \to \pi_0^* \qquad \text{(Dirac sea..)}$$

- "Intuitive" representation of $\pi_E \equiv \pi_{W_E}$

$\xi_E = f_1 \wedge f_2 \wedge \ldots$

orthonormal basis of EK

$$\begin{cases} \pi_E\,(a(f)^*)\,\xi_E = 0 & \forall f \in EK \\ \pi_E\,(a(f^\perp))\,\xi_E = 0 & f^\perp \in (1-E)K \end{cases}$$

★ Powers - Størmer's theorem

$$\pi_E \cong \pi_F \iff E - F \in \text{H.S.}$$

unitary equivalence

Hilbert Schmidt

CONSTRUCTION OF Det

Let $H = H_+ \oplus H_-$ be a complex, separable, polarized Hilbert space (typical example $H = L^2(S^1, \mathbb{C})$ with Fourier polarization) $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= E_+ - E_-$ polarization operator

$J^2 = I$. Starting from this, one has the Segal-Wilson Grassmannian, which admits the following characterization

(—, Valli '94)

$$\bigstar \quad Gr(H, H_+) = \{ W \text{ closed subspace of } H \; / \; E_W - E_+ \in H.S. \}$$

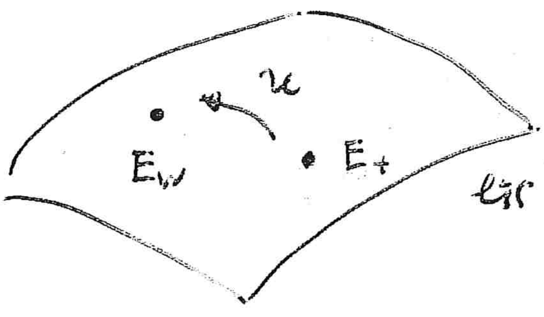
$Gr(H, H_+)$ is a $U_{res}(H)$ - homogeneous Kähler manifold, where

$$U_{res}(H) = \{ u \in \underbrace{U(H)}_{\text{unitary group}} \; / \; [u, J] \in H.S. \}$$

Any $u \in U(H)$ induces via the formula $a(f) \mapsto a(uf)$, an automorphism α_u of $A(H)$.

If $W = u E_+ u^{-1}$ one gets, by Powers-Størmer

$$\pi_{E_W} \cong \pi_{E_+} \circ \alpha_u \cong \pi_{E_+} \quad ; \; u \in U_{res}(H)$$



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$\Rightarrow \exists \ \tilde{u} \in P\,U(H_{E_+})$
projective unitary

such that $\quad u_{g_1} u_{g_2} = c(g_1, g_2)\, u_{g_1 g_2}$

$$\boxed{\pi_{E_W} = \tilde{u} \circ \pi_{E_+} \circ \tilde{u}^{-1}}$$

central extension of $U_{res}(H)$

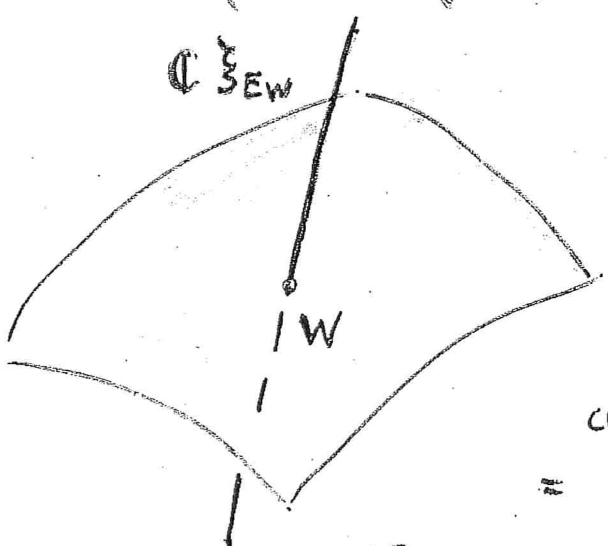
$\Rightarrow$

$$\left.\begin{array}{l} w_{E_+} \rightsquigarrow \xi_{E_+} \\ w_{E_W} \rightsquigarrow \xi_{E_W} \end{array}\right\} \in H_{E_+}$$

vectors in the same space

☆ The lines $\mathbb{C}\,\xi_{E_W}$ $\quad W \in Gr(H, H_+)$

yield the fibres of Det



one gets local triviality and equivariance.

$\rightsquigarrow$ G.Q.

curvature = p.b. F-S

= Schwinger term

TOOLS II

- Clifford algebra $\mathcal{C}(H)$

complex C^*-algebra generated by
$\varphi(f)$, $f \in H = H_{\mathbb{R}}$ real Hilbert space
subject to the relations

$$[\varphi(f), \varphi(g)]_+ = (f, g)_H I$$

- Clifford algebra $C(H_{\mathbb{C}}, B)$

$H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ complexification

$B = (\ ,\ )^{\mathbb{C}}$ complex bilinear form extending $(\ ,\ )$

The choice of a complex structure on $H_{\mathbb{R}}$: $J : H_{\mathbb{R}} \to H_{\mathbb{R}}$. $J^2 = -I$
gives rise to an isotropic subspace W of $H_{\mathbb{C}}$ (with respect to B) and to a C^*-algebra isomorphism

$$C(H_{\mathbb{C}}, B) \cong A(W)$$

$$H_{\mathbb{C}} \cong H \oplus \overline{H}$$

$$\| \quad W \oplus \overline{W} \quad \| \quad Eig(J^{\mathbb{C}}, -i)$$

$$\| \quad Eig(J^{\mathbb{C}}, +i)$$

$$\varphi(f) = \frac{1}{\sqrt{2}} \left(a(f) + a(f)^* \right)$$

$$a(f) = \frac{1}{\sqrt{2}} \left(\varphi(f) - i \varphi(Jf) \right)$$

- Action of $O(H)$
  orthogonal group

$O(H)$ acts on $C(H)$ and hence on

$C(H_{\mathbb{C}}, B) \cong A(W)$ via $*$-automorphisms

$\alpha_O \quad (O \in O(H))$ "Bogolubov transformations"

- unitary implementation of α_O

when :

$$\pi_{I_W}^{A(W)} \circ \alpha_O \cong \pi_{I_W}^{A(W)} \quad ? \quad \text{(say)}$$

that is

$$\exists\ \pi(\alpha_0(a)) = \tilde{O}\cdot\pi(a)\cdot\tilde{O}^{-1}$$

$$\tilde{O} \in PU(H_\pi)$$

?

The answer is provided by the

Shale - Stinespring theorem

which asserts (suitably reformulated)

that this is possible iff

$$\underset{\substack{\| \\ J_W}}{[O, J]} \in H.S. \qquad \left(\overset{\Leftrightarrow}{O \in O_{res}(H)} \right)$$

If $Y = O.W$ $\pi^{A(W)}_{I_W} \circ \alpha_0$ corresponds to

$\pi^{A(Y)}_{I_Y}$

this is equivalent to $J_Y - J_W \in H.S.$

- Fock states

The possible Fock states are parametrized by the Siegel manifold $\mathcal{Y}(H) = O(H)/U(H)$

transformation leaving a fixed complex structure fixed

consisting of the complex structures of $H = H_{1/2}$. This follows from

$$\omega_0(\varphi(f)\varphi(g)) = \langle f, g \rangle_J = (f, g) + i(f, Jg)$$

two-point functions

- the polarized case

Let $H = H_+ \oplus H_-$

and regard it as a real Hilbert space

we have
$$H_{\mathbb{C}} = W \oplus \overline{W} \cong H \oplus \overline{H}$$

with $W = H_+ \oplus \overline{H}_-$

CONSTRUCTION OF Pf

One considers

$$\mathcal{Y}_{res}(H) = {O_{res}(H)} \big/ {U(H)}$$

+ restricted Siegel manifold
 or isotropic Grassmannian

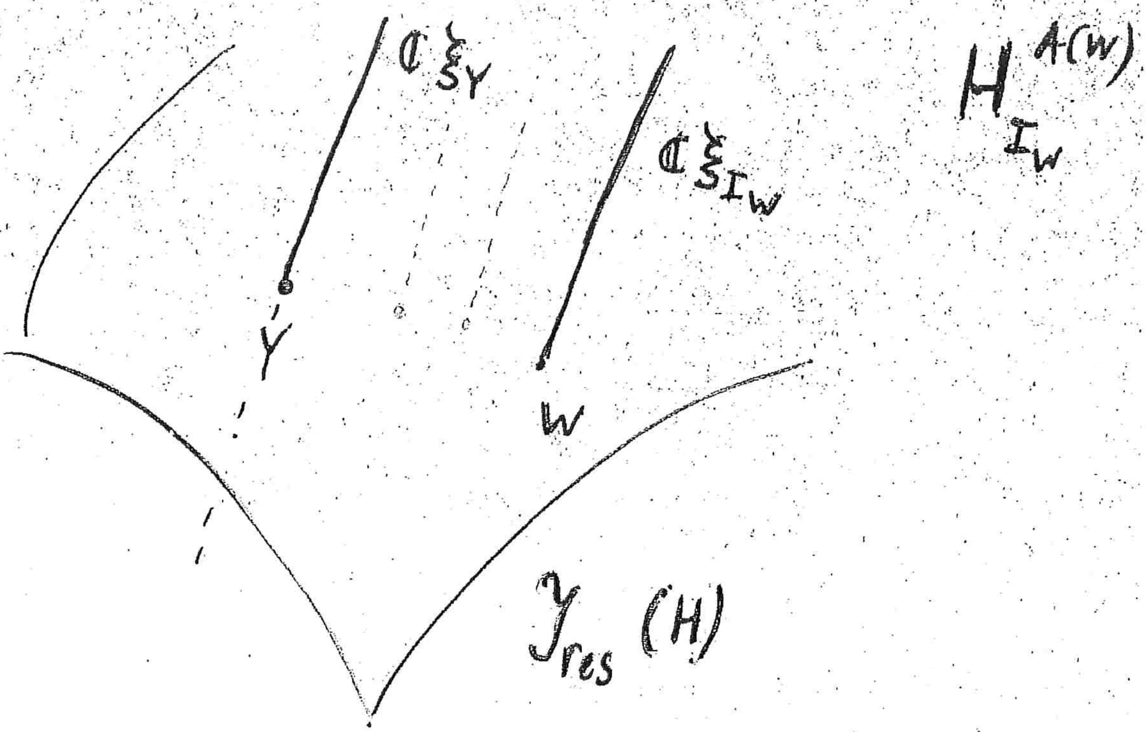
By virtue of the Shale-Stinespring theorem, the possible anti-Fock states (in the restricted class) become vector states of $H_{I_W}^{A(W)}$

$$W \rightsquigarrow w_{I_W} \rightsquigarrow \mathbb{C}\,\xi_{I_W} \in H_{I_W}^{A(W)}$$

$$OW = Y \rightsquigarrow w_{I_W} \circ \alpha_O \rightsquigarrow \mathbb{C}\,\xi_Y$$

$$\begin{matrix} \mathbb{C}\,\xi_Y \equiv Pf_Y \\ \| \\ \tilde{O} \cdot Pf_W \end{matrix}$$

fibre of the Pfaffian line bundle over $\mathcal{Y}_{res}(H)$



PROOF OF

$$\mathrm{Det}\Big|_{\mathcal{Y}_{res}(H)} = Pf^{\otimes 2}$$

via Powers-Størmer purification and the Fock-anti-Fock correspondence purification

$$\omega_{I_W} \rightarrow \omega_{I_W \oplus \bar{O}_W} = \omega_{E_W}$$

$A(W) \simeq A(W \oplus O) \hookrightarrow A(H_{\mathbb{C}})$

$$\pi_{I_W}(A(W)) \hat{\otimes} \pi_{\bar{O}_W}(A(\bar{W})) \cong \pi_{E_W}(A(H_{\mathbb{C}}))$$

$$\cong \quad \text{Fock-anti-Fock}$$

⚠ caveat

$$\pi_{I_W}(A(W))$$

$$\pi_{I_W}(A(W)) \hat{\otimes} \pi_{I_W}(A(W)) \cong \pi_{E_W}(A(H_{\mathbb{C}}))$$

$$\mathcal{H}^{A(W)}_{I_W} \hat{\otimes} \mathcal{H}^{A(W)}_{I_W} \cong \mathcal{H}^{A(H_{\mathbb{C}})}_{E_W}$$

$$\Downarrow$$

☆ $$\underset{Y}{\mathrm{Pf}_W} \otimes \underset{Y}{\mathrm{Pf}_W} = \underset{Y}{\mathrm{Det}_W}$$

by equivariance

BOREL - WEIL

$$\mathcal{S} := H^{A(W)}_{I_W} = \Gamma^2_{\mathcal{O}}(Pf^*)$$

spinors (bosonization)

realizes, à la Borel-Weil, the Spin^c representation of $\widetilde{O_{res}(H)}$, central extension of $O_{res}(H)$

PLÜCKER EMBEDDING

$$Gr(K, K_+) \hookrightarrow P(H) = H^{A(K)}_{E_+}$$

$w \in W$

$\bar{W} \in Gr(K, K_+)$ $\qquad \bar{W} \sim [w_1, w_2, \dots]$ orthonormal basis

$$a(w)^* \bar{W} = 0$$

PAULI EXCLUSION PRINCIPLE

$\mathcal{S} : \{ S \subseteq \mathbb{Z} \mid S - \mathbb{N}$ and $\mathbb{N} - S$ are finite $\}$

$$H = \ell^2(\mathcal{S}) \simeq H_{E_+}$$

$$K_S = \overline{[\, z^i \,,\, i \in S \,]}$$

$$\pi_S(W) = \langle W, K_S \rangle$$

Plücker coordinates

Representation

$$a(z^i)^* K_S = \varepsilon(i, S)\, K_{S \cup \{i\}}$$

$$\varepsilon = \begin{cases} \pm 1 & \\ 0 & i \in S \end{cases}$$

SEGRE EMBEDDING

$$S : Gr(H, H_+) \times Gr(H, H_+) \hookrightarrow Gr(H_{\mathbb{C}}, W)$$

$$(W_1 \, , \, W_2) \longmapsto W_1 \oplus \overline{W_2^{\perp}}$$

explicitly

$$\pi_{(S, T')} (W_1 \oplus \overline{W_2^{\perp}}) = \pi_S (W_1) \, \pi_T (W_2)$$

Plücker coordinates

Recall that, in finite dimensions

$$\mathbb{P}^r \times \mathbb{P}^s \hookrightarrow \mathbb{P}^{rs+r+s}$$

$$((s_i) \, , (w_i)) \longmapsto (s_i w_j)$$

$$Gr(H, H_+) \hookrightarrow \mathcal{J}_{res}(H) \hookrightarrow Gr(H_{\mathbb{C}}, W)$$

$$Pf \big|_{Gr(H, H_+)} = \text{Det} \, ..$$