GEOMETRIC METHODS IN QUANTUM MECHANICS

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Lecture XII : The HOMFLYPT polynomial via helicity and GQ

International Doctoral Program in Science

Brescia, Capitolium

SYMPLECTIC GEOMETRIC ASPECTS OF THE HOMFLYPT POLYNOMIAL

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(joint work with
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Outline

1. Symplectic geometry & geometric quantization
2. Lagrangian submanifolds & Maslov theory
3. Knot theoretic background & the HOMFLYPT polynomial
4. Geometry of Brylinski's manifold
5. Chern-Simons generating function & Maslov cycle
6. The Homflypt polynomial as a WKB-wave function ⟵

Geometric quantization

a quick sketch

- (M, ω) symplectic manifold — classical phase space
  ω closed ($d\omega = 0$), nondegenerate

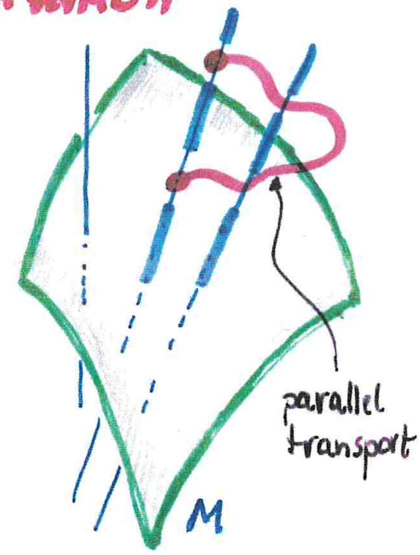
Darboux: locally $\omega = dp \wedge dq$
$= d\vartheta \equiv d(p\,dq)$

ϑ : symplectic potential

- $[\omega] \in H^2(M, \mathbb{Z})$ integrality condition

for coadjoint orbits: integrality

$\Rightarrow$ (Weil-Kostant) $\nleftrightarrow$ prequantization



$\exists \ (L, \nabla, h)$

line bundle over M — connection compatible with h — hermitian metric

s.t. $\Omega_\nabla = -2\pi i\, \omega$

curvature of ∇

$c_1(L) = [\omega]$

locally $-2\pi i \vartheta \equiv$ connection form

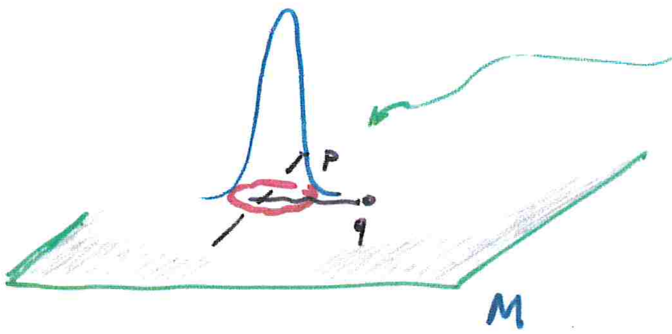
- prequantum Hilbert space: $\Gamma(L)^{-L^2}$

L^2- completion of $\Gamma(L)$

with respect to h — smooth sections of $L \to M$

$\Rightarrow$ too big!

example : the harmonic oscillator



arbitrarily localized wave functions are **forbidden** by the

* Heisenberg Uncertainty principle

need of a polarization

*** complex**

holomorphic sections

if M is Kähler situations above are automatically ruled out...

oscillator case:

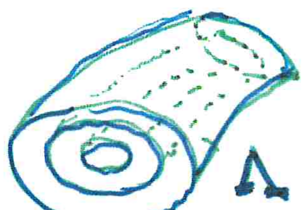
Bargmann - Fock representation

tool for (simple) Lie group representation theory:

Borel-Weil

*** real**

in particular, in the **integrable** case



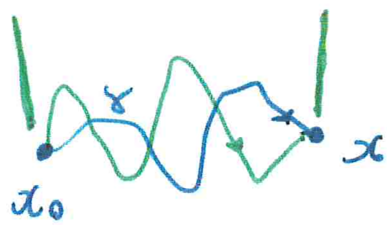
$\psi = \psi(q)$

look for **covariantly constant sections of the flat connection** $\nabla|_{\Lambda}$

$\nabla \psi = 0$

S: action

$\psi = \psi_0 \, e^{i\overset{\sim}{S_\gamma}\,\vartheta}$



* $\boxed{\text{semiclassical wave function}}$

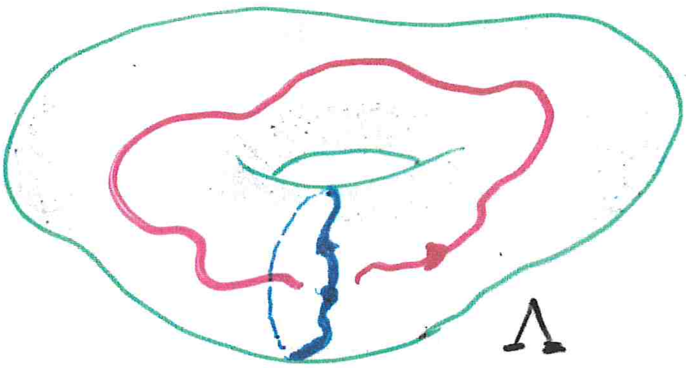
$\Rightarrow$

$$\int_\gamma \vartheta \in 2\pi \mathbb{Z}$$

trivial holonomy closed loop

$\ast$ Bohr - Sommerfeld

without Maslov's correction



Λ

$\ast$ Maslov

$$\frac{1}{2\pi}\int_{\gamma_i} \vartheta = n_i + \nu_i \qquad \nu_i = \frac{\beta_i}{4} \qquad \beta_i \in \mathbb{Z}$$

"multivaluedness of the semiclassical wave function" Keller, Maslov

• MASLOV THEORY

$\Lambda \subset T^*M$ Lagrangian submanifold
symplectic
typical situation by Weinstein's theorem

$\boxed{\text{generating function (Morse family)}}$ $\varphi = \varphi(x, \vartheta)$ $(x,\vartheta) \in M \times \mathbb{R}^k$

(local in general) auxiliary parameters

$$C_\varphi = \{ (x, \vartheta) \in M \times \mathbb{R}^k \mid d_\vartheta \varphi = 0 \} \quad \text{(control)}$$

$d(d_\vartheta)$ maximal rank

non degenerate Hessian

$$C_\varphi \to T^*M$$
$$(x, \vartheta) \mapsto (x, d_x\varphi) \in \Lambda$$

immersion with image Λ

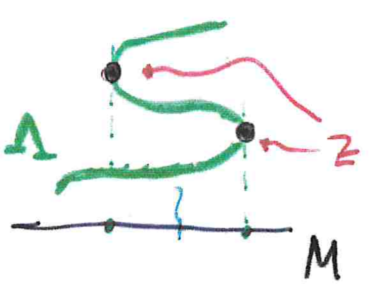
$F = F(x) := \varphi(x, \vartheta(x))$ phase function

$(x, dF(x)) \in \Lambda$ graph

momentum

★ This fails at the singular points of the projection

$\Lambda \to M$



★ singular locus Z Maslov cycle

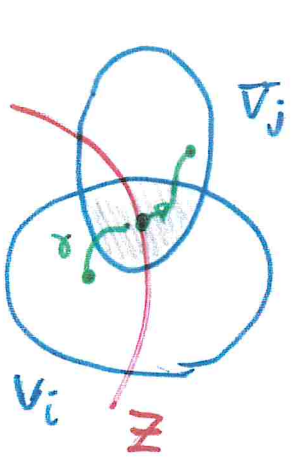
$\operatorname{codim} \partial Z \geq 3$ $\quad$ $\operatorname{codim} Z = 1$

(caustic)

given a good open cover $\{V_i\}$ of Λ

let $\left[\sigma_i = \underbrace{\operatorname{sign}}_{\text{signature}} \underbrace{\mathcal{H}_\vartheta}_{\text{Hessian with respect to } \vartheta} \Big|_{V_i \setminus Z} \right]$

1



$$\boxed{\frac{1}{2}\left(\sigma_j - \sigma_i\right) = \pm 1 = \gamma \circ Z}$$

intersection index

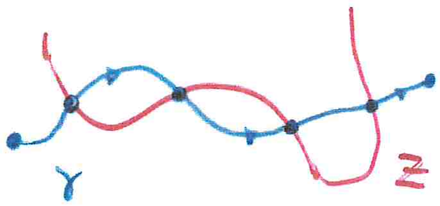
in general

$$m(\gamma) = \gamma \circ Z = \int_\gamma \eta_Z$$

Maslov index of γ

sum with appropriate signs over $\gamma \cap Z$

Poincaré dual of Z



$$\mu_{\alpha\beta} := \operatorname{sgn} \mathcal{H}_\beta - \operatorname{sgn} \mathcal{H}_\alpha$$

Maslov cocycle

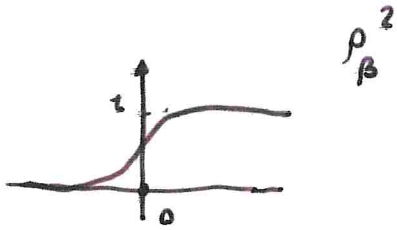
$\downarrow$? weil

$$\boxed{\mu_Z = \mu_{\alpha\beta}\, \rho_\alpha\, d\rho_\beta}$$

Poincaré dual

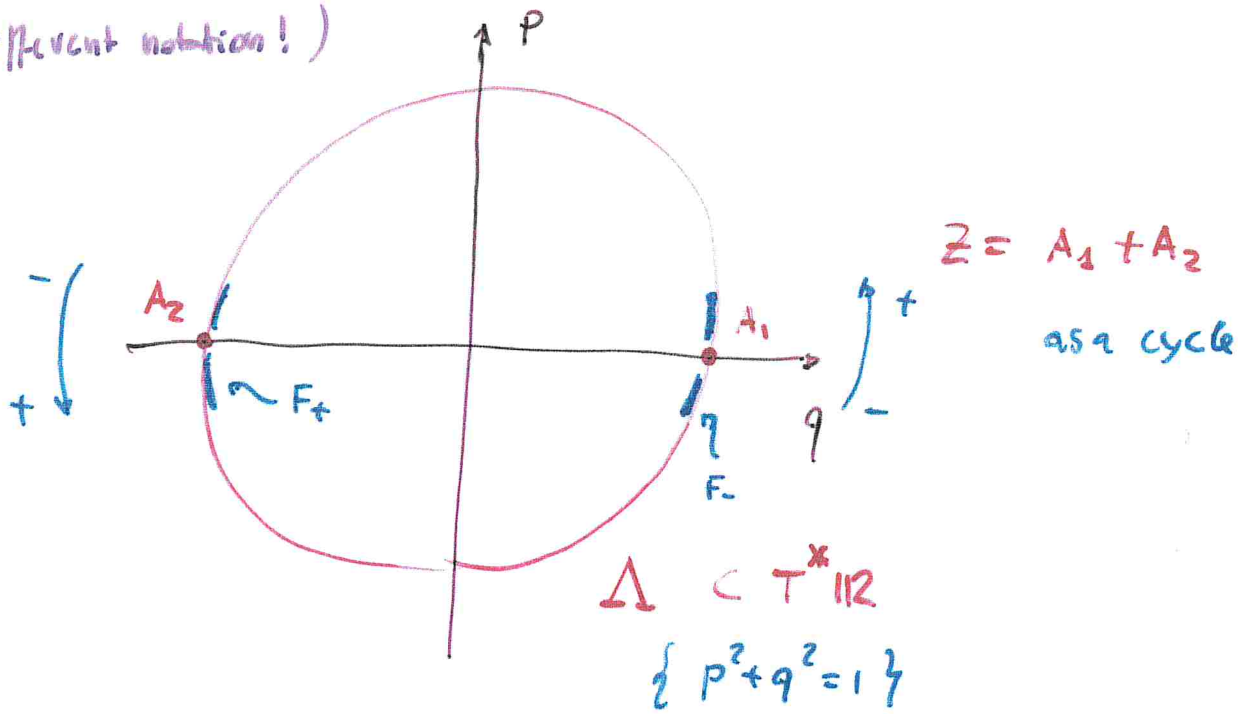
$$\mu_{\alpha\beta}\, \rho_\alpha\, d\rho_\beta = \mu_{\alpha\beta}(1-\rho_\beta)\, d\rho_\beta \sim -\mu_{\alpha\beta}\, d(1-\rho_\beta)\, \rho_\beta$$

$$= +\mu_{\alpha\beta}\, \rho_\beta\, d\rho_\beta = \frac{1}{2} \mu_{\alpha\beta}\, d\rho_\beta^2$$



Example

(different notation!)



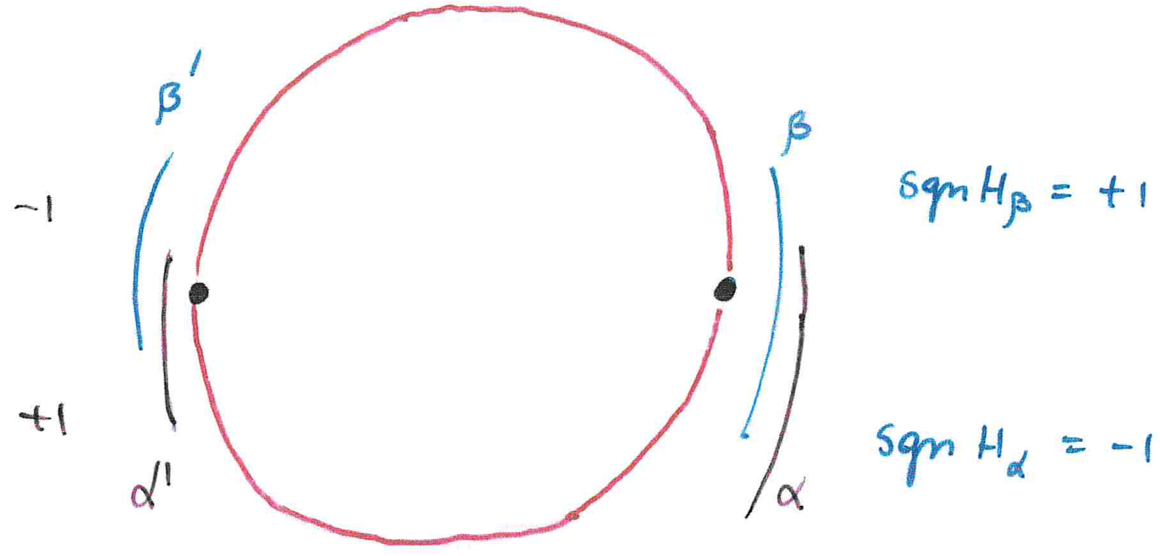
$$F_\pm(q,x) = qx \pm \int_0^x (1-t^2)^{\frac{1}{2}}\,dt \qquad \text{Morse family}$$

$$\frac{\partial F_\pm}{\partial x} = q \pm (1-x^2)^{\frac{1}{2}} = 0 \qquad q = \mp(1-x^2)^{\frac{1}{2}}$$

$$p = \frac{\partial F_\pm}{\partial q} = x \qquad \begin{cases} q = \mp(1-x^2)^{\frac{1}{2}} \\ p = x \end{cases}$$

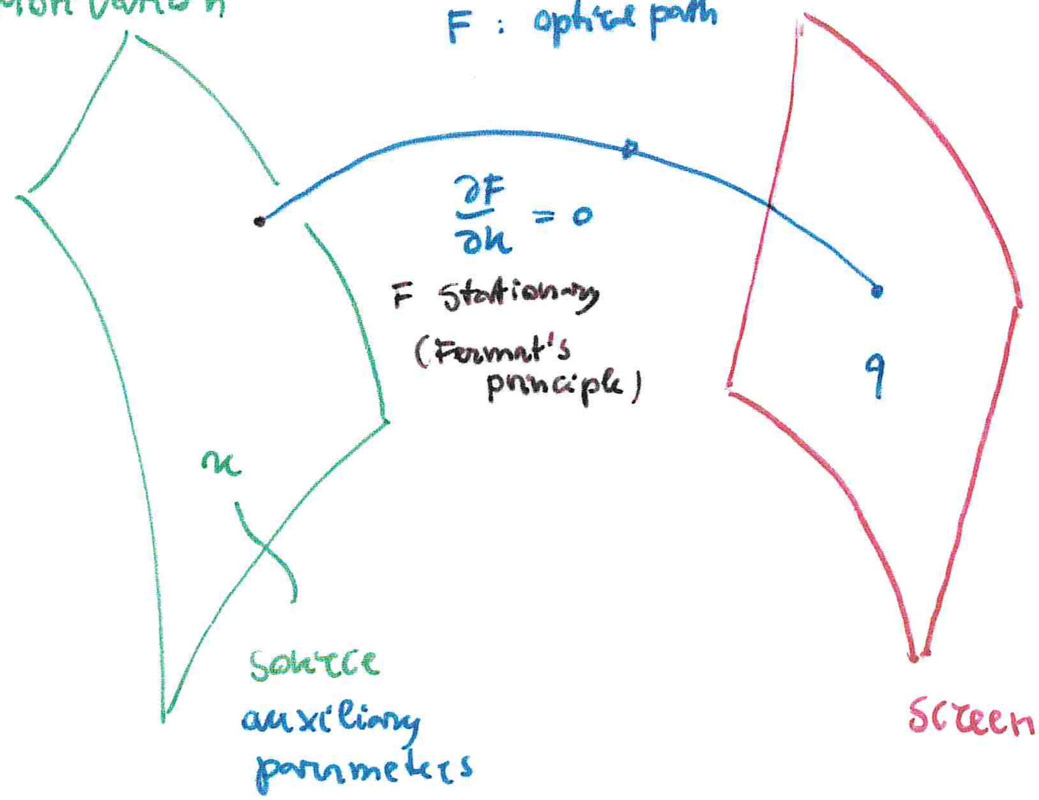
$$\frac{\partial^2 F_\pm}{\partial x^2} = \pm\frac{d}{dx}(1-x^2)^{\frac{1}{2}} = \pm\frac{1}{2}(1-x^2)^{\frac{1}{2}}(-2x)$$

$$= \mp(1-x^2)^{\frac{1}{2}}x = \frac{p}{q} \qquad q\neq 0$$

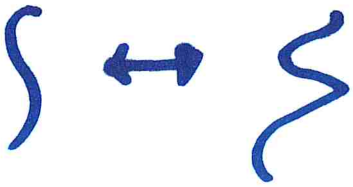


$$\mu_{\alpha\beta} = \dots = 2$$

Motivation



REIDEMEISTER MOVES



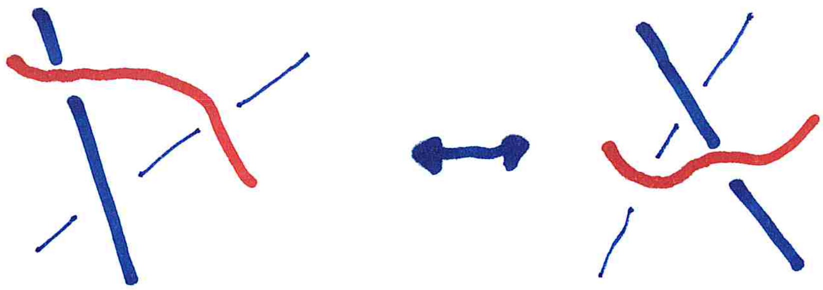
R_0

plane isotopies

R_1

(writhe changes)

R_2



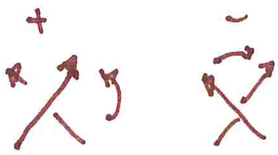
R_3

- **Regular** isotopy invariant $\tilde{I}$: under R_0, R_2, R_3
- **Ambient** isotopy invariant I: under R_0, R_1, R_2, R_3

$$\boxed{I = (-\alpha)^{w(\cdot)}\, \tilde{I}}$$

"Kauffman's principle"

★ $\boxed{w(L) = \sum_{\substack{\text{all} \\ \text{crossings}}} \pm 1}$

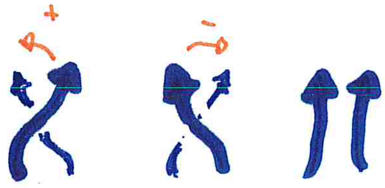


writhe

☆ THE HOMFLYPT POLYNOMIAL $P(\alpha, z)$

ambient isotopy invariant

skein relation



L_+ L_- L_0

$$\boxed{\alpha P(L_+) - \alpha^{-1} P(L_-) = z\, P(L_0)}$$

$P(O) = 1$

P computable via the skein-template algorithm

$\alpha = 1$: Conway

$z = \alpha^{-\frac{1}{2}} - \alpha^{\frac{1}{2}}$: Jones

$$\boxed{P(L) = \alpha^{-w(L)} H(L)}$$

"Kauffman's principle"

H-polynomial (regular isotopy invariant)

$H(L_+) - H(L_-) = z H(L_0)$

$H(O) = 1$

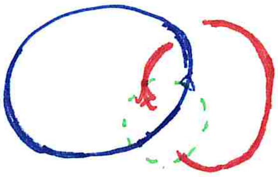
$H($  $) = \alpha H($  $)$

$H($  $) = \alpha^{-1} H($  $)$

Notice that

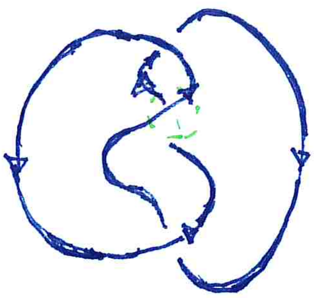
$m_\pm$, m_0 : # components of $L_\pm$, L_0

$$m_+ = m_- = m_0 \pm 1$$

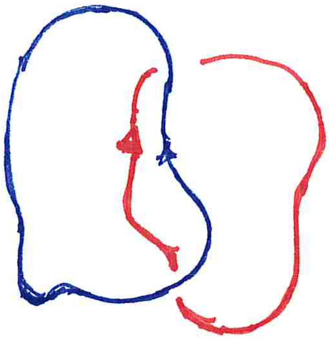


L_+

L_0

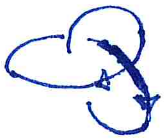


L_+



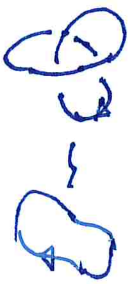
L_0

$L_\pm$ & L_0 can be mutually inequivalent

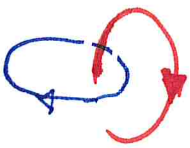


L_+

trefoil



L_- unknot



L_0 Hopf link

but

$L_+ = E_+$ $L_- = E_-$ L_0

"figures of eight"

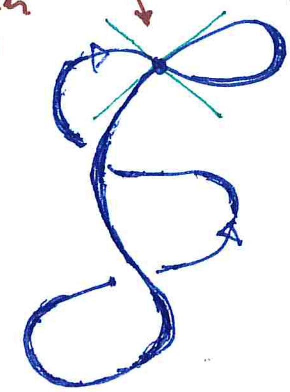
important for the sequel

Brylinski's manifold

$\hat{Y}_M$: unparametrized, oriented mildly singular knots (or links) in a 3-fold M

$\leadsto M = \mathbb{R}^3$

finite points finite order frequency

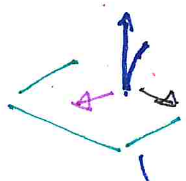


$\hat{Y}_M$

- symplectic

$\beta = \int \nu$

volume form



$J = \underline{t} \times \cdot$

- formally Kähler

K or L $\in \hat{Y}$

component: coadjoint orbit of $\mathrm{Sdiff}(M)$

$\to$ prequantizable if $[\nu] \in H^3(M, \mathbb{Z})$

$\leadsto [\beta] \in H^2(\mathcal{L}M, \mathbb{Z})$

descends to $\hat{Y}_M$

no topological obstructions for $\mathbb{R}^3$

Y_M : smooth knots (or links)

Three **Lagrangian submanifolds** involved

Λ : links on a plane

$\Lambda_\bullet$: "cones"

$\tilde{\Lambda} \subset T^*\hat{Y}$

stemming from an observation of Brylinski

$\leadsto$ same **Morse family**

via **Chern-Simons** theory

- GENERATING FUNCTIONS

$T^*\hat{Y}_{\mathbb{R}^3}$ cotangent space pertaining to $\hat{Y}_{\mathbb{R}^3}$

$\mathcal{A} = \Omega_{\mathbb{R}}(\mathbb{R}^3) \otimes \mathbb{R}^3$ compactly supported real vector fields on $\mathbb{R}^3$

auxiliary parameters

Morse family (generating function)

$$\boxed{\Phi(K,A) := \frac{\kappa}{8\pi}\underbrace{\int_{\mathbb{R}^3} A\wedge dA}_{\text{Chern-Simons action}} + \underbrace{\int_K A}_{T_K A} = \int_{\mathbb{R}^3} A\wedge\eta_K}$$

$-i\log$ (holonomy of A around K)

Singular Poincaré dual 2-form (for K a knot)

★ Helicity $\rightsquigarrow$ $\int_{\mathbb{R}^3}\vec{A}\cdot\vec{B} = \int_{\mathbb{R}^3}\vec{A}\cdot \operatorname{curl}\vec{A}$

$\Downarrow$

$$\boxed{0 = d_{\mathcal{A}}\Phi\big|_{(K,A)} = \frac{\kappa}{4\pi}\underbrace{dA}_{F_A} + T_K}$$

Curvature of A

$$\begin{cases} dA_K = -\frac{4\pi}{\kappa}T_K & \text{curvature concentrated on } K \\ \delta A_K = 0 & \text{Coulomb gauge condition} \end{cases}$$

$\operatorname{div}\vec{A} = 0$

$$\boxed{A_K = -\frac{4\pi}{\kappa}\Delta^{-1}\delta T_K}$$

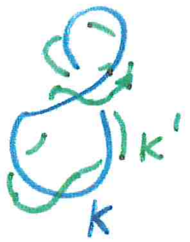
Hörmander-Łojasiewicz

Hodge Laplacian $\Delta = d\delta + \delta d$

$\in \mathcal{A}$

in order to substitute onto Φ we have to consider K ordinary and then regularize via framing

$\Downarrow$



$$\boxed{\varphi(K) = -\frac{2\pi}{k}\,\ell\underbrace{(K,K')}_{\text{linking number}}}$$

$$P\Big|_K = d_{\hat{Y}_{\mathbb{R}^3}}\varphi\Big|_{K \in Y_{\mathbb{R}^3}} = 0$$

momentum

upon choosing a plane projection & blackboard framing

$$\boxed{w = w(K)}$$

locally constant phase function (eikonal equation)

$$\Rightarrow \boxed{W = e^{-\frac{2\pi}{k} i w}}$$

"WKB-wave function"

Witten (- Kauffman)

Compute the Hessian w.r. to $\mathcal{A}$

$\Rightarrow$ get $H(\cdot,\cdot) = d_{\mathcal{A}}$

non degenerate quadratic form . but needs regularization (e.g. via ζ-function)

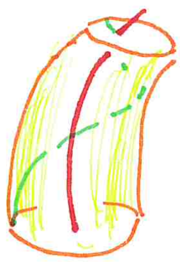
but (Atiyah), we take directly

$$H(A_K, A_K) = \mathcal{H}(K)$$

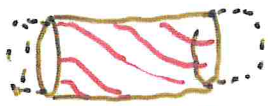
helicity of a solenoidal field associated to K

$\mathcal{A} \ni$

writhe, see below.



$\mathcal{H} = \ell k$



flux tube

Coming back to $(\hat{Y}_{\mathbb{R}^3}, \beta_0)$

$$\beta_0\big|_K = \int_K \nu_0$$

$\nu_0 = \frac{3}{2}\, dx \wedge dy \wedge dz$

$= d\mu_0 \qquad \mu_0 = \frac{1}{2}\{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy\}$

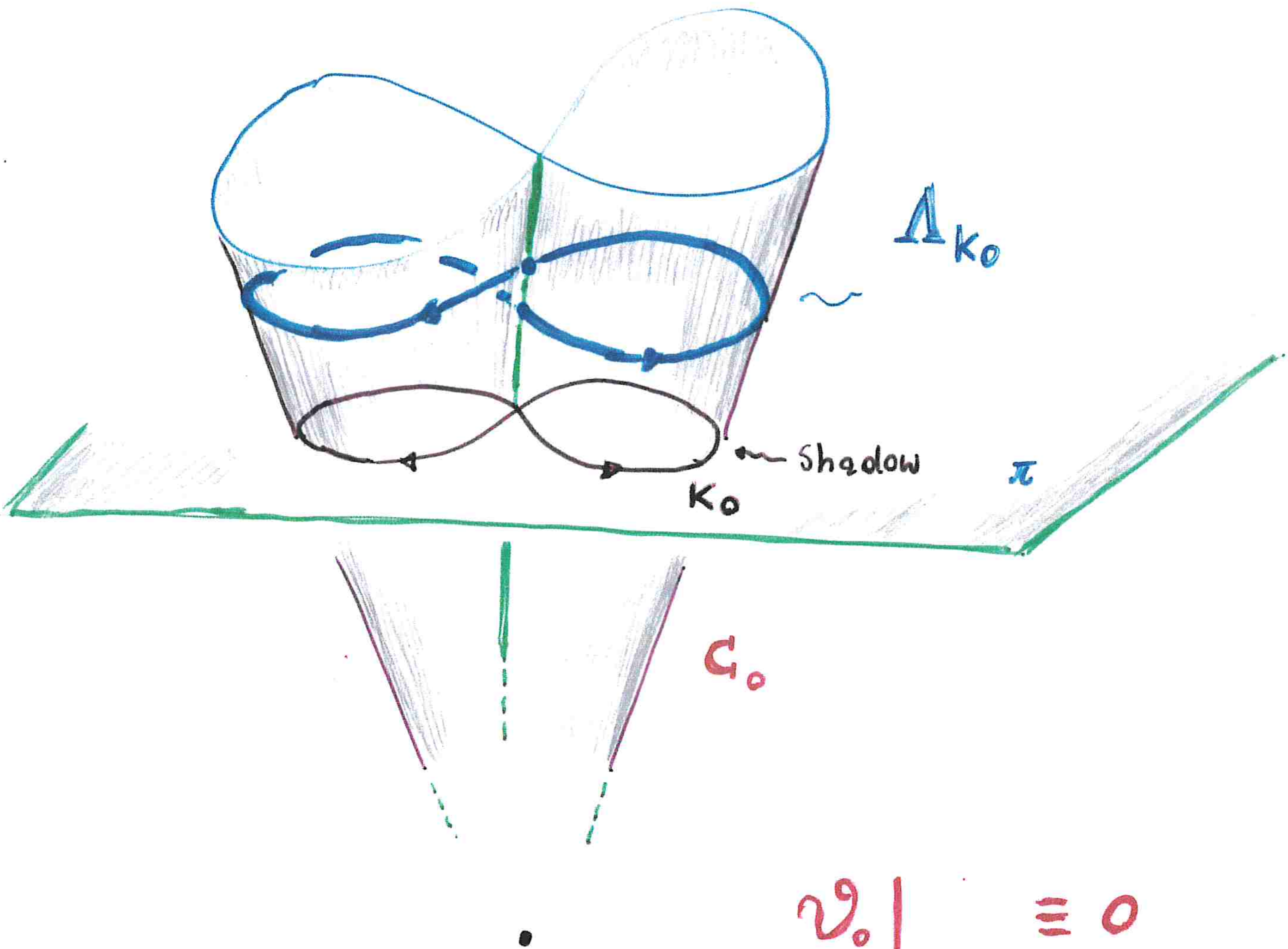
$$\vartheta_0\big|_K(\cdot) = -\frac{1}{2}\int_K \langle \mathbf{r}, \dot{\mathbf{r}} \times \cdot \rangle$$

parametrization of K

★ symplectic potential :

$$\beta_0 = d\vartheta_0$$

The Cone Construction



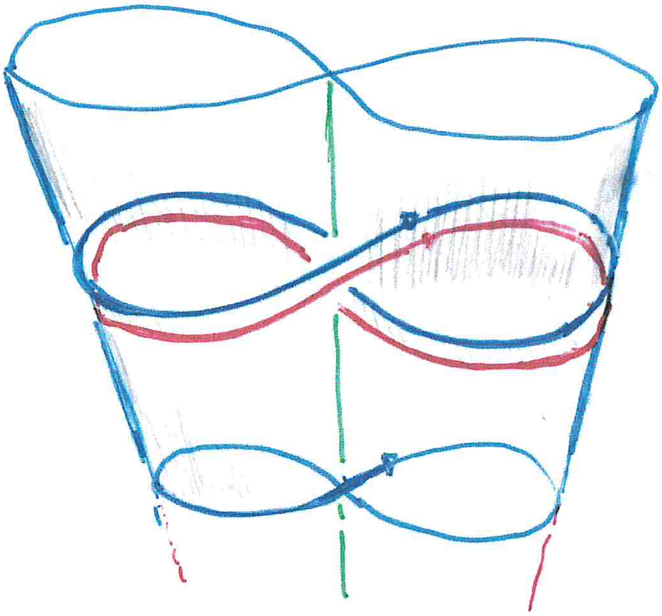
$$\vartheta_0\big|_{\Lambda_{K_0}} \equiv 0$$

$\Lambda_{K_0} = \{$ (singular) knots on C_0 above π intersecting all generating lines once, transversally $\}$
i.e. regularly projecting onto K_0

Lagrangian submanifold of $\hat{Y}_{\mathbb{R}^3}$
(modelled on $C^\infty(S^1, \mathbb{R})$)

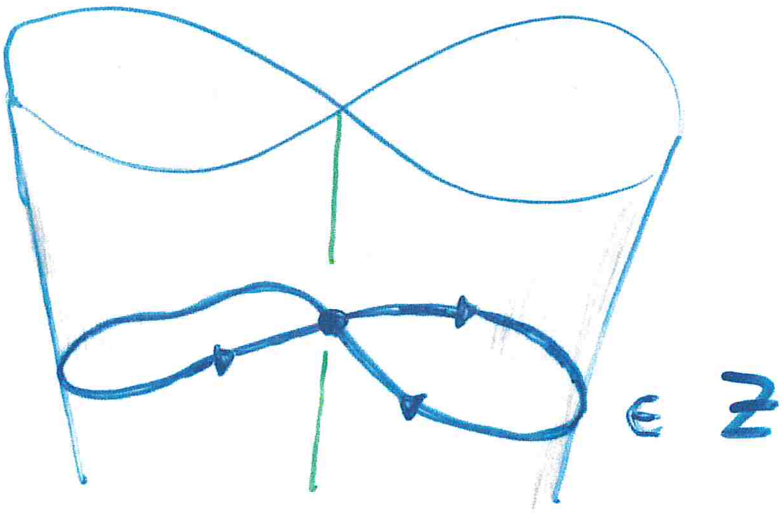
Φ still a Morse function for Λ_{K_0}

(cf. Weinstein...)



~ blackboard framing

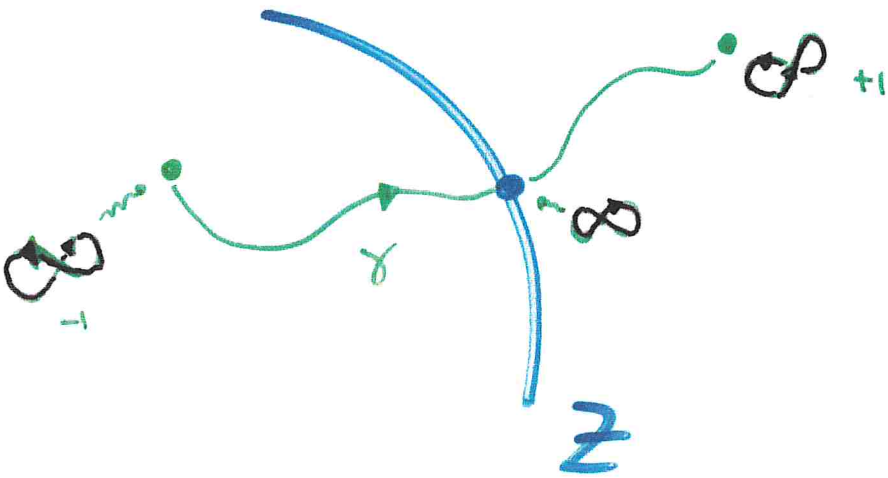
writhe

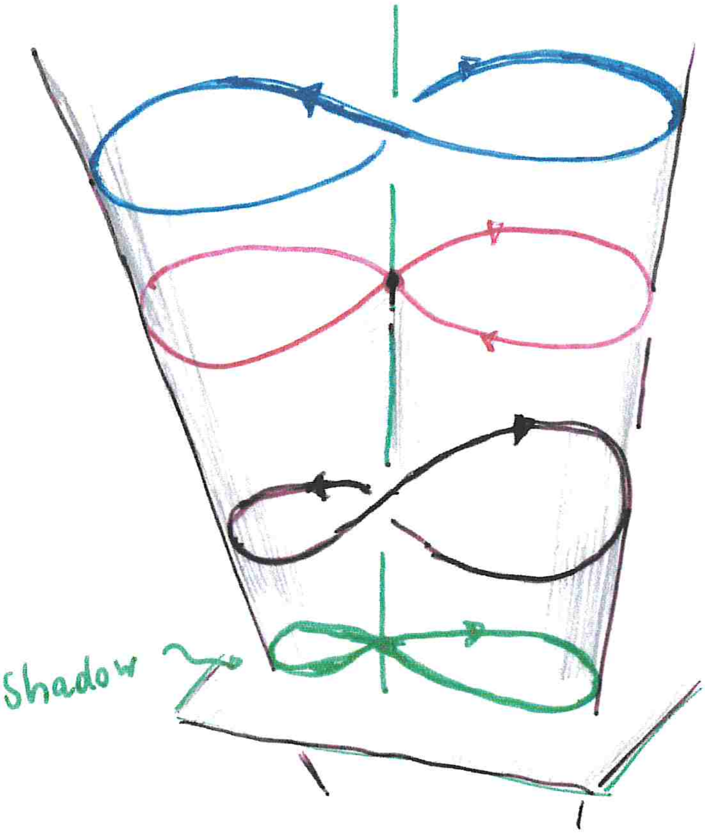


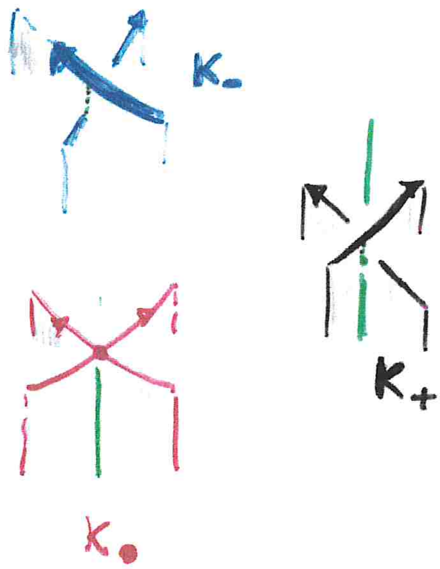
Maslov cycle

natural co-orientation

in general: exactly one singular point







$$\frac{1}{2}\left(w(K_+) - w(K_-)\right) = 1 = \gamma \circ Z$$

via M-R

$$\frac{1}{2}\left(\mathcal{H}(K_+) - \mathcal{H}(K_-)\right) = 1 = \gamma \circ Z$$

intersection index

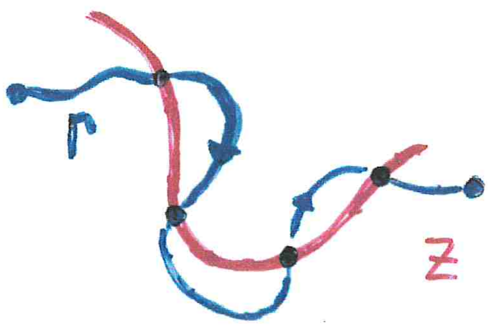
$$\boxed{\text{signature} \underset{def}{=} \text{helicity} \underset{MR}{=} \text{writhe}}$$

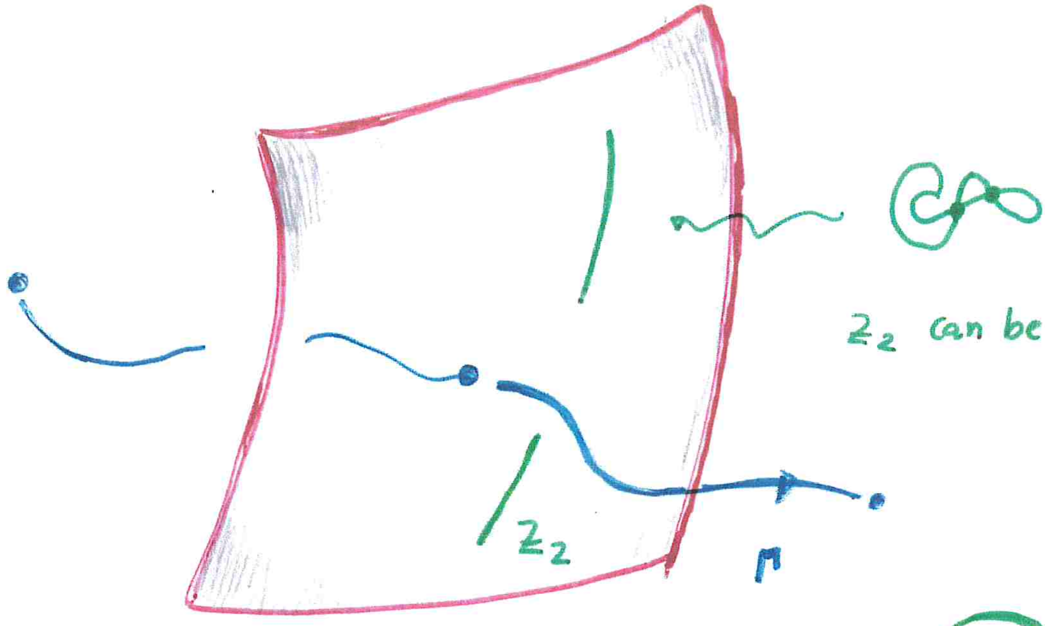
$$m(\Gamma) = \Gamma \circ Z$$

path of knots
(crossing just $Z \equiv Z_1$)

$m(\Gamma) \equiv 0$
for a
closed
path...

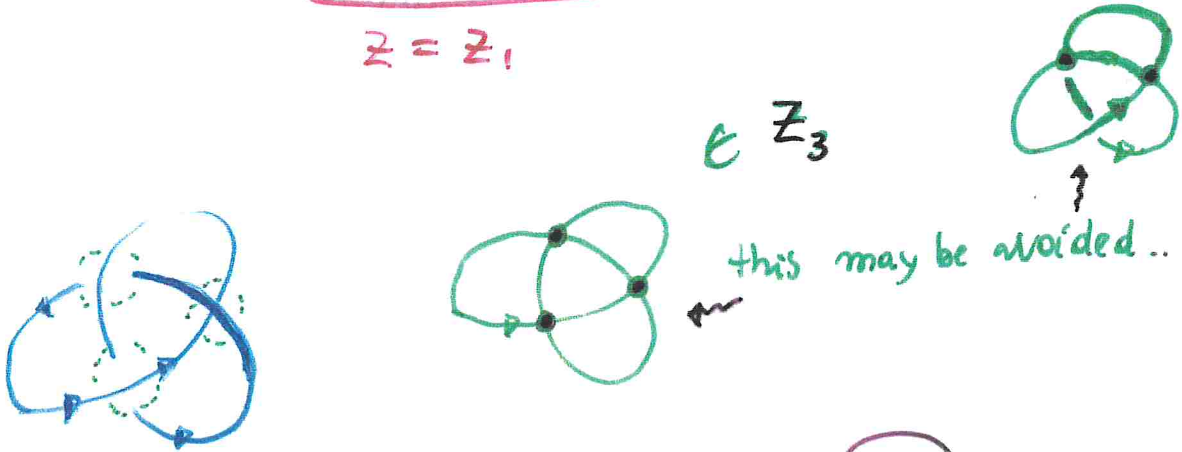
no global
Maslov index





Z_2 can be avoided

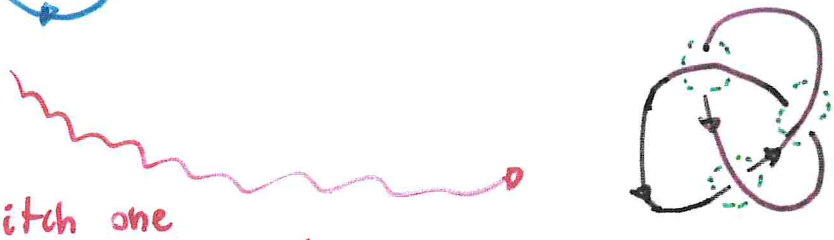
$z = z_1$



$\in Z_2$

$\in Z_3$

this may be avoided..



all crossings changed

switch one crossing at a time...

(traversal of z ...)

$$\boxed{dw =: 2\,\eta_z = p}$$

enhanced eikonal formula

singular Poincaré dual of z

generalized momentum

cf. geometric optics

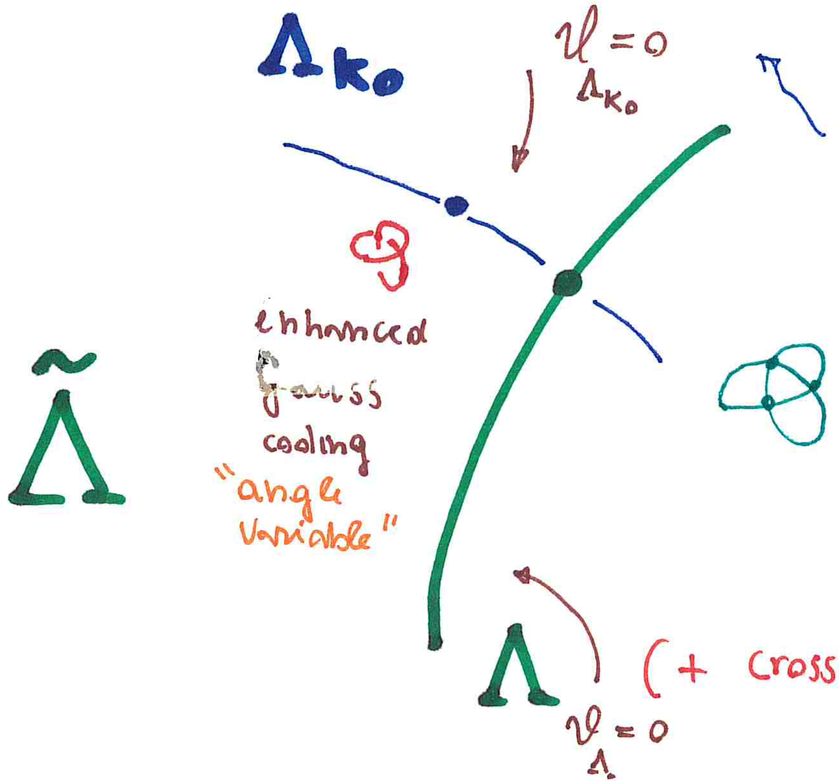
sudden phase switches across caustics

Φ : Morse family

for all objects

locally trivial fibration

canonical section : trivial knot



Purely **Maslov** phenomena intervene

Knot shadow (Gauss coding)
"action variable"

(+ crossing information)

$$\Psi(K) = \alpha^{W(K)} = e^{i\lambda W(K)}$$

Witten's regular isotopy invariant :

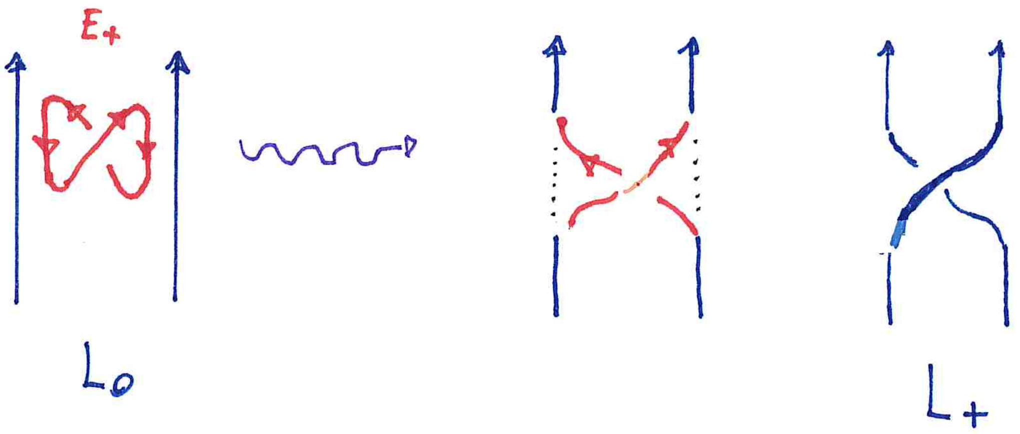
WKB wave function

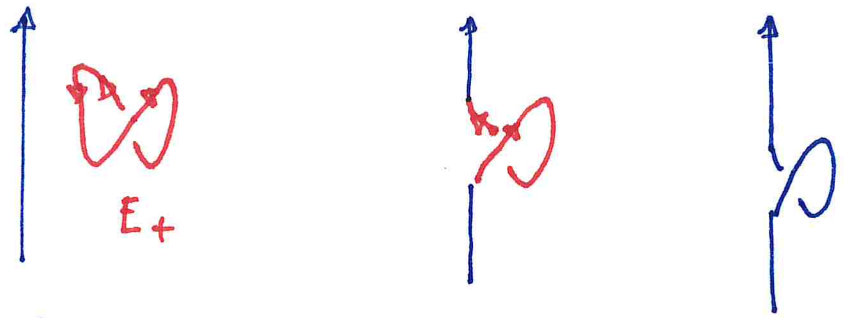
covariantly constant section

$$\boxed{\Psi(L) = \alpha^{W(L)} = \alpha^{\mathcal{H}(L)} = e^{i\lambda \mathcal{H}(L)}}$$

$\mathcal{L} \to \Lambda$ trivial bundle

★ $E_\pm$ - SURGERY   inspired by Liu - Ricca

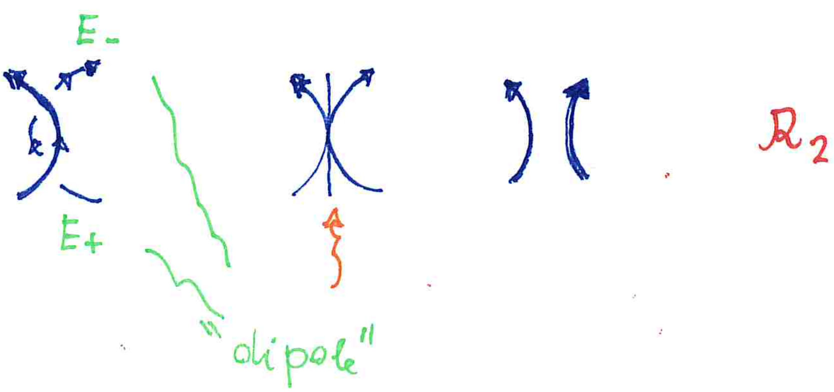




single strand

R_1
(first Reidemeister move)

Also notice



★ CRUCIAL STEP

$$\psi(L) = \alpha^{\mathcal{H}(L)} = e^{i\gamma \mathcal{H}(L)} = \alpha^{w(L)}$$

'' L_0

Abelian regular isotopy invariant

$$\alpha\,\psi(L) - \alpha^{-1}\psi(L) =$$

$$\underbrace{\alpha\,\psi(L_0)}_{\substack{\| \\ \psi(L_+)}} - \underbrace{\alpha^{-1}\psi(L_0)}_{\substack{\| \\ \psi(L_-)}} = (\alpha - \alpha^{-1})\,\psi(L_0)$$

$E_\pm$-surgery

$$\boxed{\begin{aligned} &\psi(L_+) - \psi(L_-) = (\alpha - \alpha^{-1})\,\psi(L_0) \\ &\psi(\ ) = \alpha\,\psi(\ ) \\ &\psi(\ ) = \alpha^{-1}\,\psi(\ ) \end{aligned}}$$

Ansatz ★

z arbitrary

$$\boxed{\begin{aligned} &\tilde{\psi}(L_+) - \tilde{\psi}(L_-) = z\,\tilde{\psi}(L_0) \\ &\tilde{\psi}(\ ) = \alpha\,\tilde{\psi}(\ ) \\ &\tilde{\psi}(\ ) = \alpha^{-1}\,\tilde{\psi}(\ ) \end{aligned}}$$

$\rightsquigarrow$ H - polynomial

$$\Psi(L) := \alpha^{-w(L)}\,\tilde{\Psi}(L)$$

Ψ $\rightsquigarrow$ **HOMFLYPT** $\searrow$ Conway, Jones

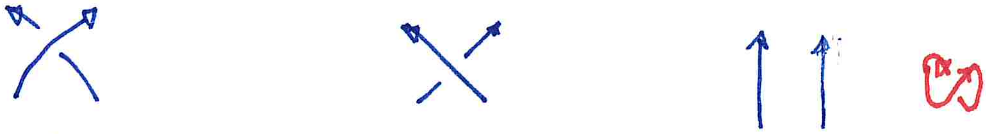
WKB - wave function

$$\boxed{\begin{array}{c} \alpha\,\Psi(L_+) - \alpha^{-1}\Psi(L_-) = z\,\Psi(L_0) \\ \Psi(\bigcirc) = 1 \end{array}}$$

$\square$

not present in the knot case

$$\Psi(L_+) = \underbrace{\alpha^{-2}}_{\text{Maslov}}\Psi(L_-) + \underbrace{z\,\alpha^{-1}}_{E_+\text{-surgery}}\Psi(L_0)$$



L_+ $\qquad$ L_- $\qquad$ L_0

cf Liu - Ricca : " $P = t^{\mathfrak{H}}$ "

$L_\pm$ & L_0 may lie in mutually different components

L_0 has a different number of components from $L_\pm$