V2

Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture XLVIII

GOOD COVERINGS
MAYER-VIETORIS CALCULATIONS

✱ good covers (or coverings) of a manifold

Def. Given a topological space X and a cover (open) $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in\Omega}$, $X = \bigcup_{\alpha\in\Omega} \mathcal{U}_\alpha$, $\mathcal{U}$ is called good if all finite intersections

$$\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2} \cap \ldots \cap \mathcal{U}_{\alpha_\ell}$$

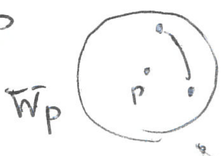
are either empty or contractible (in particular, such are the $\mathcal{U}_\alpha$ themselves...)

(for manifolds, equivalently: diffeomorphic to $\mathbb{R}^n$)

Via Riemannian geometric techniques$^+$ one proves that every smooth manifold M admits a good covering.

In particular, if M is compact, the cover can be chosen to be finite (but non compact manifolds can admit a finite good cover, e.g. $\mathbb{R}^n$). In the sequel we shall deal with manifolds admitting a finite good cover only..

(+) One endows M with a Riemannian metric g: this is always possible. The covering is produced via the Riemannian exponential map via showing existence of (open) geodesically convex neighbourhoods of any point p (i.e. any two points of $\overline{W}_p$ are joined by a unique geodesic totally lying in $\overline{W}_p$



The reason for focussing on such manifolds is the following

+ **Theorem**

The coohomology of smooth manifolds having a finite good cover is finite-dimensional (i.e. $h^q(M) = \dim H^q(M) < \infty \quad \forall q$)

Proof. The proof can be easily achieved by resorting to the Poincaré lemma, to the MV sequence and by induction on the (finite) cardinality of the covering. More specifically:

$$M = \bigcup_{\alpha\in\mathcal{J}} \mathcal{U}_\alpha \qquad H^q(\mathcal{U}_\alpha) \cong H^q(\mathbb{R}^n) = \begin{cases} \mathbb{R} & , q=0 \\ \{0\} & , q>0 \end{cases}$$

finite set

$$\mathcal{U}_\alpha \underset{\text{diffeom.}}{\approx} \mathbb{R}^n \qquad H^q(\mathcal{U}_{\alpha_0}\cap\cdots\mathcal{U}_{\alpha_k}) \cong \mathbb{R} \text{ or } \{0\}$$

The MV-sequence (a chunk thereof.) reads
for two open sets

$$\cdots\to \underset{\substack{\uparrow \\ \text{finite dimensional}}}{H^{q-1}(U\cap V)} \xrightarrow{d^*} H^q(U\cup V) \xrightarrow{r} \underbrace{H^q(U)}_{\text{finite}} \oplus \underbrace{H^q(V)}_{\text{dimensional}} \to\cdots$$

hence:

$$h^q(U\cup V) \underset{N+R}{=} \dim\ker r + \overbrace{\dim \operatorname{Im} r}^{<\infty}$$

$$= \underbrace{\dim \operatorname{Im} d^*}_{<\infty} + \overbrace{\dim \operatorname{Im} r}^{<\infty}$$

$\Rightarrow H^q(U\cup V)$ is finite-dimensional.

$\square$

★ Computation of cohomology groups via MV

$\boxed{H^*(\mathbb{R}^2 \setminus \{p,q\})}$ ↙ two points removed

$M = \mathbb{R}^2 \setminus \{p,q\}$

$U = \mathbb{R}^2 \setminus \{p\} \underset{\text{omt.}}{\approx} S^1 \times \mathbb{R}$

$V = \mathbb{R}^2 \setminus \{q\} \underset{\text{omt}}{\approx} S^1 \times \mathbb{R}$

$U \cup V = \mathbb{R}^2$

one has immediately $\underset{\substack{\mathbb{R}^2\setminus\{p,q\} \\ \text{connected}}}{H^0 = \mathbb{R}}$, $H^2 = 0$ e.g. $\mathbb{R}^2 \setminus \{p,q\} \equiv M$ non compact

$$\boxed{\begin{aligned} H^0(M) &\cong \mathbb{R} \\ H^1(M) &\cong \mathbb{R}^2 \\ H^2(M) &\cong 0 \end{aligned}}$$

Let us, however, ascertain this by MV.

$\boxed{H^0}$

$$0 \dashrightarrow \underset{\substack{\mathbb{R} \\ \text{Poincaré}}}{H^0(\mathbb{R}^2)} \xrightarrow{f} \underset{\mathbb{R}}{H^0(U)} \oplus \underset{\mathbb{R}}{U(V)} \xrightarrow{g} \underset{\substack{\| \\ x}}{H^0(M)} \xrightarrow{d^*} \underset{\substack{0 \\ \text{Poincaré}}}{H^1(\mathbb{R}^2)}$$

$\mathbb{R} = \operatorname{Im} f = \boxed{\ker g}$ $\boxed{\operatorname{Im} g} = \ker d^* = H^0(M) \Rightarrow (N+R)$

$1 + h^0(M) = 2 \Rightarrow h^0(M) = 1$ ⟶ $\boxed{H^0(M) = \mathbb{R}}$

★ variant: $1 - 2 + x = 0 \Rightarrow x = 1$

$\boxed{H^1}$

$$\underset{\substack{\| \\ 0 \\ (\text{Poincaré})}}{H^1(\mathbb{R}^2)} \to \overset{\mathbb{R}}{H^1(U)} \oplus \overset{\mathbb{R}}{H^1(V)} \xrightarrow{f} \underset{\substack{\| \\ y}}{H^1(M)} \xrightarrow{d^*} \underset{\substack{\| \\ 0 \\ (\text{Poincaré})}}{H^2(\mathbb{R}^2)}$$

$\Downarrow$ $2 - y = 0$ $y = 2$

f is an isomorphism

$\Rightarrow$ $h^1(M) = 2$ $\boxed{H^1(M) = \mathbb{R}^2}$

$\boxed{H^2}$

$$\underset{\substack{\| \\ 0}}{H^2(\mathbb{R}^2)} \to \underset{\substack{\| \\ 0}}{H^2(U)} \oplus \underset{\substack{\| \\ 0}}{H^2(V)} \xrightarrow{f} H^2(M) \xrightarrow{d^*} \underset{\substack{\| \\ 0}}{H^3(\mathbb{R}^2)}$$

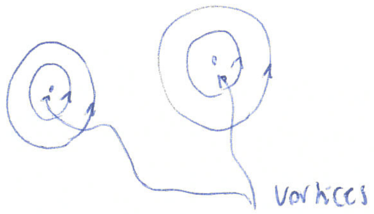
$\Rightarrow \boxed{H^2(M) = \{0\}}$

In general one gets $H^1(\mathbb{R}^2 \setminus \{P_1 \dots P_k\}) \cong \mathbb{R}^k$,

and $H^1(\mathbb{R}^2 \setminus \mathbb{Z}) \cong \mathbb{R}^\infty = \bigoplus_{\mathbb{Z}} \mathbb{R}$

 hence this manifold cannot admit a <u>finite</u> good cover.

Explicit representatives: $w_i = \dfrac{X_i \, dY_i - Y_i \, dX_i}{X_i^2 + Y_i^2}$



$X_i = x - x_{P_i}$

$Y_i = y - y_{P_i}$

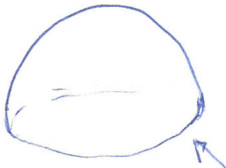
$\boxed{H^*(S^2)}$

$$\begin{cases} H^0(S^2) = \mathbb{R} \\ H^1(S^2) = 0 \\ H^2(S^2) = \mathbb{R} \end{cases}$$

Poincaré duality

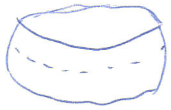
$U \cup V = S^2$

$U =$

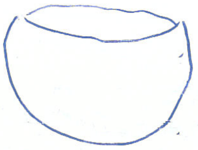


$U \cap V$

"belt"



$V =$



slightly bigger than a hemisphere

$U \cap V \underset{\text{diff}}{\approx} S^1 \times \mathbb{R}$

"equator"

4 variant

H^1: $1 - 2 + 1 + x = 0$

$\Rightarrow x = 0$

$$\begin{tikzcd}
 & S^2 & U \sqcup V & U \cap V & \\
\cdots \arrow[r, "0"] & H^0(S^2) \arrow[r] & H^0(U) \oplus H^0(V) \arrow[r, "f"] & H^0(U \cap V) \arrow[r, "d^*"] & \cdots \\
\text{clear} \arrow[r, squiggly] & \mathbb{R} & \mathbb{R} \quad \mathbb{R} & \mathbb{R} & \\
 & x & & & \\
\cdots \arrow[r, "d^*"] & H^1(S^2) \arrow[r, "g"] & H^1(U) \oplus H^1(V) \arrow[r, "h"] & H^1(U \cap V) \arrow[r, "d^*"] & {} \\
 & & 0 \quad 0 & \mathbb{R} & \\
 & y & & & \\
{} \arrow[r, "d^*"] & H^2(S^2) \arrow[r, "k"] & H^2(U) \oplus H^2(V) \arrow[r] & H^2(U \cap V) \arrow[r, dashed] & {} \\
 & & 0 \quad 0 & 0 &
\end{tikzcd}$$

4 variant

H^2:

$1 - y = 0$

$y = 1$

$\boxed{H^1}$

$\ker g = \operatorname{Im} d^*$ $\qquad$ $\ker d^* = \operatorname{Im} f = \mathbb{R}$

$\Rightarrow \operatorname{Im} d^* = \{0\}$ $\Rightarrow$ $\boxed{\ker g = \{0\}}$

$\boxed{H^1 = 0}$

$\boxed{\operatorname{Im} g = \{0\}}$ $\Rightarrow$ $H^1(S^2) \underset{N+R}{=} \{0\}$

$\boxed{H^2}$

$\ker k = H^2$ $\qquad$ $\ker k = \operatorname{Im} d^*$ $\qquad$ $\ker d^* = \operatorname{Im} h = \{0\}$

$\Rightarrow (N+R)$ $h^2(S^2) = 1$ $\Rightarrow$ $\boxed{H^2(S^2) = \mathbb{R}}$

In general

$$\boxed{H^*(S^n)} \qquad \begin{cases} H^0(S^n) = \mathbb{R} \\ H^1(S^n) = \{0\} \\ \vdots \\ H^{n-1}(S^n) = \{0\} \\ H^n(S^n) = \mathbb{R} \end{cases}$$

This result can be easily obtained by induction (from $n=1$)

(again take $S^n = U \cap V$ ← bigger than an hemisphere

$$U \cap V \underset{\text{diff}}{\approx} S^{n-1} \times \mathbb{R}$$

↖ generalised equator

$$\begin{tikzcd}
 & S^n & U \sqcup V & {\substack{S^{n-1}\times\mathbb{R} \\ U \cap V}} & \\
H^0 & \mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[r] & \mathbb{R} \arrow[r, "d^*"] & {} \\
H^1 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r, "d^*"] & {} \\
 & & \vdots & & \\
H^{n-1} & 0 \arrow[r] & 0 \arrow[r] & \mathbb{R} \arrow[r, "d^*"] & {} \\
H^n & H^n(S^n) \arrow[r] & 0 & &
\end{tikzcd}$$

$H^{n-1}(S^{n-1}\times\mathbb{R}) \simeq H^{n-1}(S^{n-1}) \cong \mathbb{R}$

inductive hypothesis

$0 \to \bullet \to ? \to 0$

or $0 \to ? \to \bullet \to 0$

$\Rightarrow \bullet \underset{\text{iso}}{\cong} ?$

$\Rightarrow H^n(S^n) \cong \mathbb{R}$

$\boxed{H^*(\mathbb{T}^2)}$

$$\begin{cases} H^0(\mathbb{T}^2) \cong \mathbb{R} \\ H^1(\mathbb{T}^2) \cong \mathbb{R}^2 \\ H^2(\mathbb{T}^2) \cong \mathbb{R} \end{cases}$$

* variant

$$\begin{tikzcd} 0 \arrow[r] & \underbrace{H^0(\mathbb{T}^2)}_{1} \arrow[r] & H^0(U)\oplus H^0(V) \arrow[d] \end{tikzcd}$$

h^1

$1 - 2 + 2 - x + 2 - 2$

$+ \underset{h^2}{y} = 0$

$y - x + 1 = 0$

$y = x - 1$

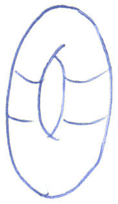
one computes e.g.

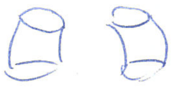
$y = 1$ (see below)

$\Rightarrow \; x = 2$

$U =$ 

$V =$ 

$U \cup V =$ 



$U \cap V$

$H^0(\mathbb{T}^2) \cong \mathbb{R}$ is left as an exercise

* let us compute H^1

$$\begin{tikzcd} \overset{\mathbb{R}^2}{H^0(U)\oplus H^0(V)} \arrow[r, "f"] & \overset{\mathbb{R}^2}{H^0(U\cap V)} \arrow[r, "g"] & \overset{x}{H^1(\mathbb{T}^2)} \arrow[r, "h"] & \overset{\mathbb{R}}{H^1(U)} \oplus \overset{\mathbb{R}}{H^1(V)} \arrow[r, "i"] & \overset{\mathbb{R}^2}{H^1(U\cap V)} \end{tikzcd}$$

$z \quad z' \longmapsto z' - z$

$\ker i \cong \mathbb{R} \Rightarrow \boxed{\operatorname{Im} h \cong \mathbb{R}}$

$\ker f \cong \mathbb{R} \Rightarrow \operatorname{Im} f \cong \mathbb{R}$ (N+R)

$\mathbb{R} \cong \operatorname{Im} f = \ker g \Rightarrow \operatorname{Im} g \cong \mathbb{R}$ (N+R)

$\mathbb{R} \cong \operatorname{Im} g = \ker h \Rightarrow \boxed{\ker h \cong \mathbb{R}} \underset{N+R}{\Rightarrow} \boxed{H^1(\mathbb{T}^2) \cong \mathbb{R}^2}$

* Let us compute H^2

$$\begin{tikzcd} \overset{\mathbb{R}^2}{H^1(U)\oplus H^1(V)} \arrow[r, "f"] & \overset{\mathbb{R}^2}{H^1(U\cap V)} \arrow[r, "g"] & H^2(\mathbb{T}^2) \arrow[r, "h"] & \underset{0}{\overset{}{H^2(U)}} \oplus \underset{0}{H^2(V)} \end{tikzcd}$$

$H^2(\mathbb{T}^2) = \ker h = \operatorname{Im} g$

$\ker g = \operatorname{Im} f \cong \mathbb{R} \underset{(N+R)}{\Rightarrow} \boxed{H^2(\mathbb{T}^2) \cong \mathbb{R}}$