V2 Lectures on
DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture L

DEGREE THEORY

★ Degree of a proper map $f: \mathbb{R}^n \to \mathbb{R}^n$

Def. A continuous map $f: \mathbb{R}^n \to \mathbb{R}^n$ is said to be proper if pre-images of compact sets are compact as well.

(Recall that the image of a compact set via a continuous map is always compact)

Example: $f: \mathbb{R} \to \mathbb{R}$, $x \longmapsto c$ $\forall x \in \mathbb{R}$ is continuous but not proper: $f^{-1}(c) = \mathbb{R}$, non compact. ($\{c\}$ is compact)

★ Proposition. If $f: \mathbb{R}^n \to \mathbb{R}^n$ is proper, then its image $f(\mathbb{R}^n)$ is closed.

Proof. Let $y \in \mathbb{R}^n$ be a limit point of $f(\mathbb{R}^n)$. Then there exists a sequence $y_k \to y$, with $y_k = f(x_k)$ for some $x_k \in \mathbb{R}^n$, for all $k = 1, 2 ..$

Let $K := \{y_k\}_{k=1,2..} \cup \{y\}$. K is obviously compact

$\Rightarrow$ $f^{-1}(K)$ is compact (since f is proper).

Now, $f(f^{-1}(K))$ is compact (f is continuous) and contains $\{y_k\}$, hence y, namely, $\exists x \in \mathbb{R}^n$ such that $y = f(x)$. $\square$

Now assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and proper. Then the following map is well-defined:

$$f^* : H_c^n(\mathbb{R}^n) \longrightarrow H_c^n(\mathbb{R}^n) \cong \mathbb{R}$$

compactly supported cohomology

via: $[\alpha] \longmapsto [f^*\alpha]$

where α represents a <u>generator</u> of $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$

It can be taken as a bump n-form with $\int_{\mathbb{R}^n} \alpha = 1$

one has

$$[f^*\alpha] = \deg f \cdot [\alpha]$$

where $\deg f$ is the <u>degree</u> <u>of f</u>, and it is given by

$$\deg f := \int_{\mathbb{R}^n} f^*\alpha \quad \in \mathbb{R}$$

(<u>a</u> <u>priori</u>)

compactly supported

The important point is that

$$\boxed{\deg f \in \mathbb{Z}} \quad (*)$$

In order to prove $(*)$ we need a digression

<u>Aside</u>:

$$H_c^*(\mathbb{R}^n) = \begin{cases} 0 & q=0 \\ 0 & \\ \vdots & \\ 0 & \\ \mathbb{R} & q=n \end{cases}$$

In general one has

$$\boxed{H^q(M) \cong H_c^{n-q}(M)^*}$$

see forthcoming chapters

If both cohomologies are <u>finite</u> <u>dimensional</u>, one has $H^q(M)^* \cong H_c^{n-q}(M)$

and indeed $H^q(M) \cong H_c^{n-q}(M)$

consequences:

① M <u>orientable</u>, <u>connected</u>, $\partial M = \emptyset$
$\Rightarrow H_c^n(M) \cong \mathbb{R}$
($\Rightarrow$ if M is compact one has $H^n(M) \cong \mathbb{R}$)

Indeed $H^0(M) \cong \mathbb{R}$
$\Rightarrow H_c^n(M)^* \cong \mathbb{R}$
$\Rightarrow H_c^n(M) \cong \mathbb{R}$

② Under the same assumptions, if M is non compact, then $H^n(M) \cong 0$

Indeed $H^n(M) \cong H_c^0(M)^*$
but $H_c^0(M) \cong 0$ (obvious: $df=0 \Rightarrow f=c$, which, for $c \neq 0$ is not compactly supported).
Hence $H^n(M) \cong 0$

Def. ① Given $f: \mathbb{R}^m \to \mathbb{R}^n$, smooth, $\boxed{m \le n}$

$p \in \mathbb{R}^m$ is a critical point for f if

(*) $f_*|_p$ is not injective ("f is not immersive at p")

i.e. $\operatorname{Ker} f_*|_p$ is non trivial

② $\boxed{m \ge n}$ In this case, p is said to be critical for f if

(**) $f_*|_p$ is not surjective "f is not submersive at p"

Notice that, by the N+R theorem

$$m = \underbrace{\dim \operatorname{Ker} f_*|_p}_{\nu(f)_p} + \underbrace{\dim \operatorname{Im} f_*|_p}_{r(f)_p}$$

if $r(f)_p < n$ ($f_*|_p$ not surjective)

$$\nu(f)_p = m - r(f)_p \ge n - r(f)_p > 0$$

So $f_*|_p$ is not injective as well.

|| Clearly, for $n = m$ $f_*|_p$ is surjective if and only if it is injective.

If p is a critical point, then $f(p)$ is called a critical value. The set of critical values is denoted by $C(f)$. The regular values constitute the complementary set $\mathbb{R}^n \setminus C(f)$.

The following result is crucial, however we shall not prove it

* Sard's lemma Let $f : \mathbb{R}^m \to \mathbb{R}^n$ (for all m, n)

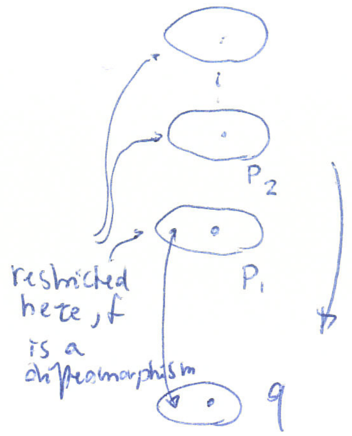
$C(f)$ has (Lebesgue) measure zero.

* Proposition Let $f : \mathbb{R}^n \to \mathbb{R}^n$ proper and not surjective. Then $\deg f = 0$

Proof. Let $p \notin f(\mathbb{R}^n)$. Since $f(\mathbb{R}^n)$ is closed, there exists a neighbourhood $U \ni p$ such that $U \cap f(\mathbb{R}^n) = \phi$. Then take a bump n-form α with support in U: obviously $f^*\alpha = 0$, so $\deg f = 0$ $(\in \mathbb{Z})$.

Now take $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth, proper and surjective. By Sard's lemma, almost all values of f are regular, so take one of them, say q.

Since $f_*|_q$ is an isomorphism, by the inverse function theorem, f is locally a diffeomorphism that is, around any $p \in f^{-1}(q)$, which is finite, being both a discrete and compact set (f is proper)



(f is a local covering map).

Choose $\alpha \in \Lambda^n(\mathbb{R}^n)$ with support in a suitable neighbourhood of q

then

$$\deg f = \int_{\mathbb{R}^n} f^*\alpha = \sum_{f^{-1}(q)} (\pm 1)$$

according to orientation

$$\Rightarrow \qquad \deg f \in \mathbb{Z}$$

(and one checks that it does not depend on the choice of the regular value q).

The definition of deg can be given for n-dimensional compact, oriented manifolds M, N. Let w be a generator for $\underbrace{H^n(N)}_{\cong \mathbb{R}}$ $\left(\int_N w = 1\right)$. Define:

$$\deg f = \int_M f^* w \quad \in \mathbb{Z} \quad \text{(again)}$$

Since Sard's lemma holds for manifolds as well, and again $\deg f = \sum_{f^{-1}(q)} (\pm 1)$, q regular value

★ Important example (related to complex analysis)

$$f : S^1 \to S^1 \quad \text{smooth} \quad (\text{automatically proper})$$

$$\boxed{\deg f = \frac{1}{2\pi}\int_{S^1} f^* d\varphi \in \mathbb{Z}}$$

angular form

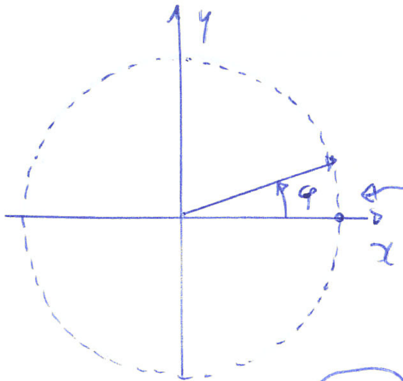
notational abuse:
$\psi = f(\varphi)$
$d\psi = \frac{df}{d\varphi} d\varphi \equiv f^* d\varphi$

Every integer can be obtained: take $f_m : e^{i\varphi} \mapsto e^{im\varphi}$, $m \in \mathbb{Z}$

$$\deg f_m = m \cdot \frac{1}{2\pi}\int_0^{2\pi} d\varphi = m \qquad \psi = m\varphi \equiv f_m(\varphi)$$

Actually every map $f : S^1 \to S^1$ which is base point preserving $\qquad S^1 \cong \{|z|=1\}$

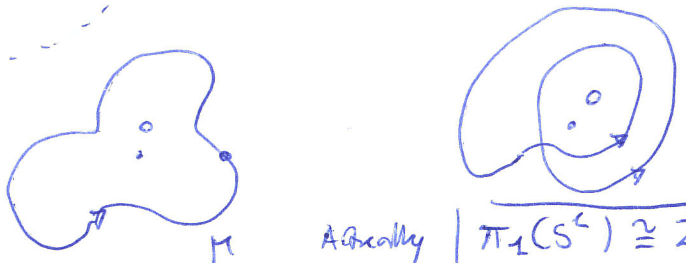
take $z=1$



is homotopic to some f_m

notice, in fact, that $\deg f$, being an integer, is constant on homotopy classes

Take

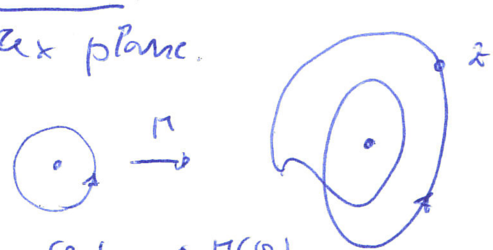


Actually $\boxed{\pi_1(S^1) \cong \mathbb{Z}}$

continuous closed curve Γ in the complex plane.

★ The winding number of Γ around 0

is given by (complex logarithm)



$\varphi \longmapsto \Gamma(\varphi) \equiv z$

$$\frac{1}{2\pi i}\int_0^{2\pi} \underbrace{d \log \Gamma}_{= \frac{d\Gamma}{\Gamma}} \;=\; \deg f, \text{ if } \boxed{f = \frac{\Gamma}{\|\Gamma\|}}$$

(+) see next page

Aside: the complex logarithm

$z \neq 0 \qquad z = |z| e^{i\varphi}$

$\log z = \log|z| + i\varphi_R$

defined on a "Riemann Surface" — "multivalued function"

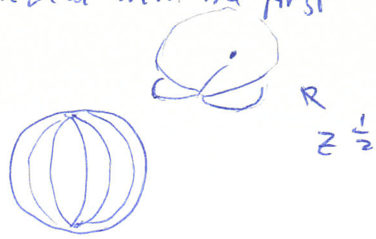
winding number (or index) of Γ around $a \notin \Gamma$

$$\mathrm{Ind}_a(\Gamma) = \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta - a}$$

The map f_m is related to m^{th} root extraction

$$0 \neq z \longmapsto z^{\frac{1}{m}}$$

$\leftarrow$ its Riemann surface consists of m sheets, the last one connected with the first



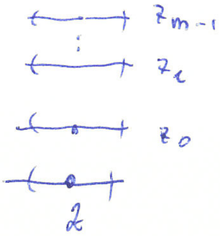
Riemann sphere picture

($z=0$ is a branching point)

m roots of z

$$f_m^{-1}(\underset{\neq 0}{z}) = \{ z_0, z_1, \ldots z_{m-1} \}$$

$$z_k = |z|^{\frac{1}{m}} \cdot e^{i(\overset{[0,2\pi)}{\varphi} + \frac{k}{m} 2\pi)}$$



$\leftarrow$ local diffeomorphisms

$$(+) \qquad d \log \Gamma = d \log \|\Gamma\| \cdot \overset{f}{\widetilde{\frac{\Gamma}{\|\Gamma\|}}} = d \log \|\Gamma\| + d \log f$$

$$d \log f = \frac{df}{f} = i \, d\psi$$

$$f = e^{i\psi} \qquad df = i e^{i\psi} d\psi = i f \, d\psi$$

$\Rightarrow$

$$\frac{1}{2\pi i} \int_0^{2\pi} d \log \Gamma = \underbrace{\frac{1}{2\pi i} \int_0^{2\pi} d \log \|\Gamma\|}_{= 0} + \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{d\varphi} d\varphi$$

$$= \deg f$$

$$\left(d\psi = \frac{d\psi}{d\varphi} d\varphi = f^* d\varphi \right)$$