V2

Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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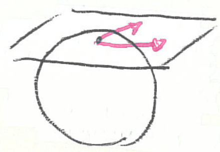
Lecture LI

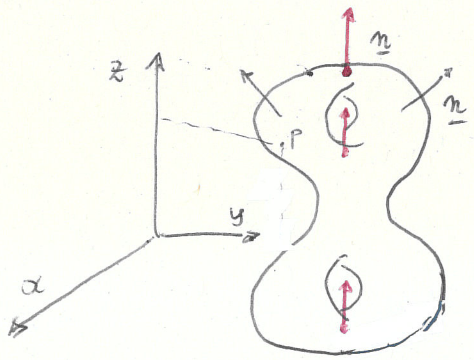
APPLICATIONS OF DEGREE: GAUSS-BONNET ; GAUSS' LINKING NUMBER

* Applications of the degree

1. The Gauss-Bonnet theorem revisited



Σ_g: closed oriented surface



Consider the Gauss map

$\langle \underline{n}, \underline{n}_u \times \underline{n}_v \rangle$
$\parallel$
$\pm \|\underline{n}_u \times \underline{n}_v\| \, \underline{n}$

$$\underline{n} : \Sigma_g \longmapsto S^2$$
$$x \longmapsto \underline{n}_x$$

$z = z(P)$: height of $P \in \Sigma_g$

$$\boxed{\deg \underline{n} = \frac{1}{4\pi} \int_{\Sigma_g} n^* \nu}$$

ν: area form on S^2

$$\int_{S^2} \frac{\nu}{4\pi} = 1$$

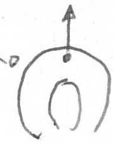
$\left[\frac{\nu}{4\pi}\right]$ generates $H^2(S^2) \cong \mathbb{R}$ (it is actually an integral form)

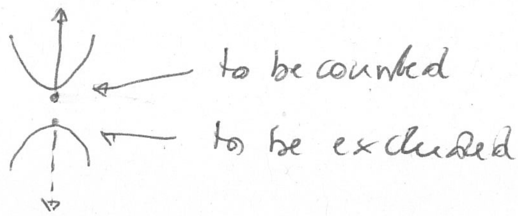
But, by the very definition of the Gauss map

$$\int_{\Sigma_g} n^* \nu = \int_{\Sigma_g} K \, d\sigma$$

$d\sigma$: area form on Σ_g

Now take  $\underline{N} \in S^2$ "north pole". $\underline{N}$ is a regular value of $\underline{n}$. Its preimage $\underline{n}^{-1}(\underline{N})$

consists of the maximum of the height function $z = z(p)$  and of half of its saddle points



Therefore $\deg \underline{n} = \sum_{\underline{n}^{-1}(\underline{N})} (\pm 1)$ counted appropriately.

Now, since the maximum is an elliptic point ($K>0$), it comes with a $+$, whereas the saddle points (hyperbolic) ($K<0$) yield a $-$.

Hence

$$\boxed{\deg \underline{n} = 1 - g}$$

(in the picture we have $g = 2$)

and we have obtained the Gauss-Bonnet theorem:

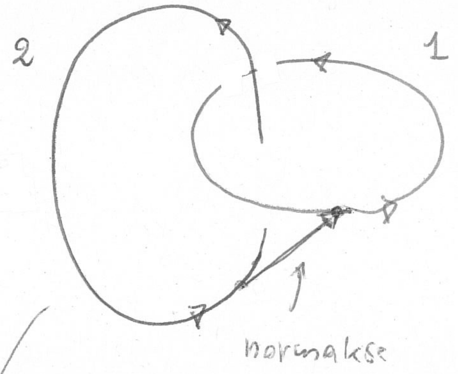
$$\boxed{\frac{1}{4\pi} \int_{\Sigma_g} K \, d\sigma = 1 - g}$$

or, equivalently

$$\boxed{\frac{1}{2\pi} \int_{\Sigma_g} K \, d\sigma = 2 - 2g = \chi(\Sigma_g)}$$

2. The <u>Gauss</u> <u>linking</u> <u>number</u> (Gauss, 1831)

Take two smooth embedded circles in $\mathbb{R}^3$ (i.e. two <u>knots</u>)
(and orient them)



$\gamma_i : S^1_i \to \mathbb{R}^3 \qquad t_i \in [0,1]$

$i = 1, 2$

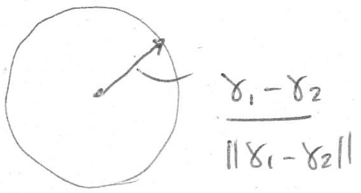
set
$\dot{\gamma}_i = \dot{\gamma}_i(t_i)$

Define the maps:

$$\boxed{\begin{aligned} f : \quad & \mathbb{T}^2 = S^1_1 \times S^1_2 \longrightarrow S^2 \\ & (t_1, t_2) \longmapsto \frac{\gamma_1(t_1) - \gamma_2(t_2)}{\| \gamma_1(t_1) - \gamma_2(t_2) \|} \end{aligned}}$$

angular variables, appropriately scaled

$f =$



Let us compute the <u>degree</u> of f

$$f = \frac{F}{\|F\|}, \qquad F(t_1, t_2) = \gamma_1(t_1) - \gamma_2(t_2)$$

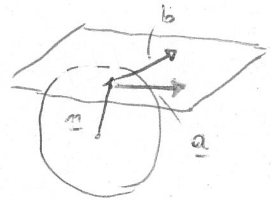
$$\deg f = \frac{1}{4\pi} \iint_{\mathbb{T}^2} f^* \nu$$

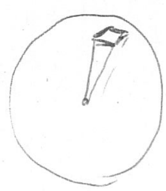
area form on S^2
oriented via the outer normal

recall:

area form: $\quad \nu_{\underline{n}}(\underline{a}, \underline{b}) = \langle \underline{n}, \underline{a} \times \underline{b} \rangle$

$\underline{a}, \underline{b} \in T_{\underline{n}}(S^2)$





Notice that, $\forall\ \underline{a}, \underline{b} \in \mathbb{R}^3$,

$$\langle \underline{n}, \underline{a} \times \underline{b} \rangle = \langle \underline{n}, \underline{a}_\perp \times \underline{b}_\perp \rangle$$

where $\underline{a}_\perp$ denotes the component of $\underline{a}$ orthogonal to $\underline{n}$: $\quad \underline{a} = \underline{a}_\perp + \langle \underline{a}, \underline{n} \rangle \underline{n}$

Now let us compute

$$f^* \nu \underbrace{(\underline{a}_1, \underline{a}_2)}_{\substack{\text{tangent vectors} \\ \text{to } T^2 \text{ (at a} \\ \text{generic point)}}} \underset{(\text{def})}{=} \nu(f_* \underline{a}_1, f_* \underline{a}_2)$$

$$= \langle f, \underbrace{f_* \underline{a}_1 \times f_* \underline{a}_2}_{\text{tangent vectors to } S^2} \rangle$$

We have, by the chain rule $\left(f = \frac{F}{\|F\|} \right)$

$$f_* \underline{a}_1 = \frac{F_*}{\|F\|} \underline{a} + \underbrace{c(\underline{a}, f)}_{\text{scalar}} f$$

$\triangle$! This part disappears within $\times$, see the preceding remark

$$\Rightarrow \nu(f_* \underline{a}_1, f_* \underline{a}_2) =$$

$$\langle F, F_* \underline{a}_1 \times F_* \underline{a}_2 \rangle .$$

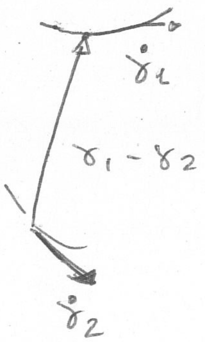
This entails that

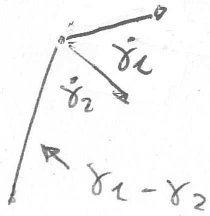
This is called the "Biot-Savart form"

$$f^* \nu = \frac{\langle \gamma_1 - \gamma_2, \dot{\gamma}_1 \times \dot{\gamma}_2 \rangle}{\|\gamma_1 - \gamma_2\|^3} dt_1 \wedge dt_2$$

notice this

1





we eventually get

$$\deg f = \frac{1}{4\pi} \int_0^1 \int_0^1 \frac{\langle \gamma_1 - \gamma_2 , \dot{\gamma}_1 \times \dot{\gamma}_2 \rangle}{\| \gamma_1 - \gamma_2 \|^3} \, dt_1 \, dt_2$$

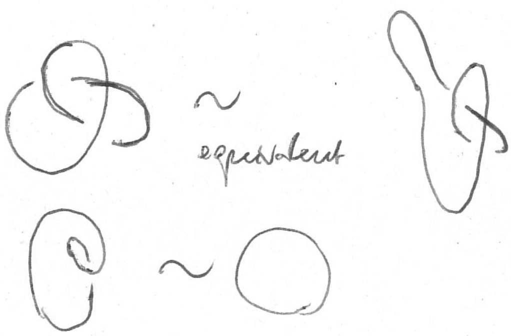
any parameter can be used..

$=: \ell(\gamma_1, \gamma_2)$ (Gauss linking number: he proved,

$(= \ell(\gamma_2, \gamma_1))$ via Ampère's law in electromagnetism, that the r. h. s. is an integer, not depending on the (ambient) isotopy class of γ_1 and γ_2.

This property is clear from the interpretation of $\ell(\gamma_1, \gamma_2)$ as the degree of a map.

A **knot**, or more generally a link is an ambient isotopy class of (smooth, w.l.o.g. to avoid pathologies) curves in $\mathbb{R}^3$ (an isotopy is a transformation of the link induced via a diffeomorphism of the ambient space $\mathbb{R}^3$)

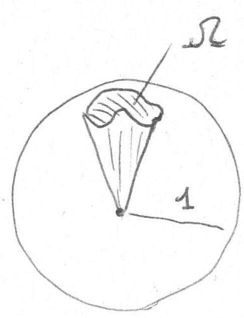


Notice that

$$\ell(\gamma_1, \gamma_2) = \frac{1}{4\pi} \iint d\Omega$$

↖ element of solid angle

Ω = area intercepted by the sphere of radius 1



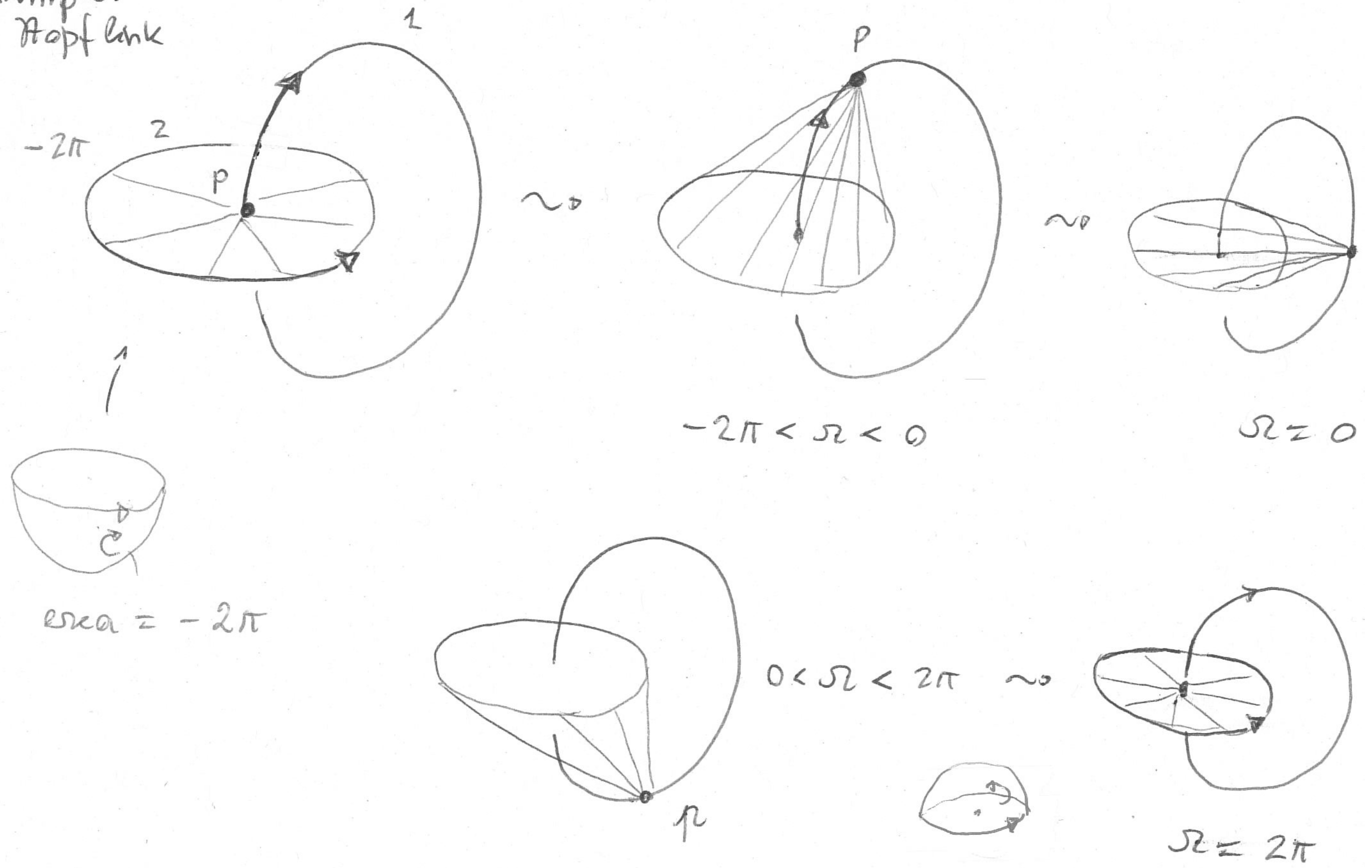
$$= \frac{\Delta\Omega}{4\pi}$$

↙ measured in steradians

← normalising factor = area of the sphere of radius 1

$$= \frac{\left\{ \begin{array}{l} \text{total variation of the solid angle obtained by taking} \\ \text{one of the curves fixed and letting the vertex of} \\ \text{the cone slide along the other one} \end{array} \right\}}{4\pi}$$

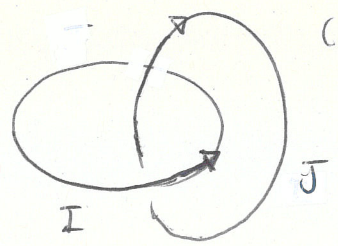
* Example: the Hopf link



total variation $\Delta\Omega = 2\pi - (-2\pi) = 4\pi$

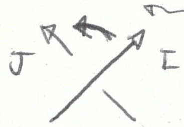
$$\Rightarrow \quad \ell(\gamma_1, \gamma_2) = +1$$

The linking number can be computed in several other ways. Here is a simple, combinatorial method:



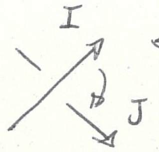
(regularly) project the link on a plane, examine the overcrossings of I over J:

if you have



← counterclockwise rotation; ⟶ assign +1

whilst, if you have



← clockwise rotation, assign −1

Thus we have the following definition

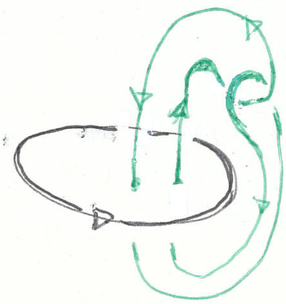
$$\boxed{\ell(I,J) = \sum_{\substack{\text{overcrossings} \\ \text{of } I \text{ over } J}} \pm 1 \qquad (= \ell(J,I))}$$

For the above Hopf link one gets **+1**. It is not difficult to conclude that it coincides with the one given by the previous definition.

For

the trivial 2-component link one has $\ell = 0$

but caveat!

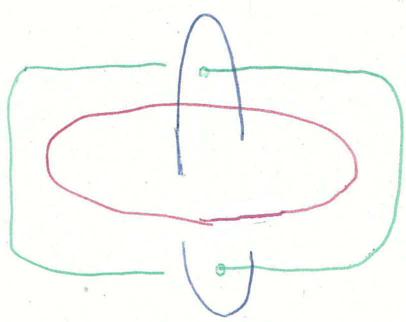


For this link (the Whitehead link) one gets $\ell = 0$, however it is non trivial

($\neq$ )

We close this digression by showing the celebrated

★★★ Borromean rings:



The linking numbers of any two components are 0, however the full link is not trivial. If you remove a component, the other two fall apart, giving a trivial link.

"Higher linking" phenomena emerge...

... but this is another story...