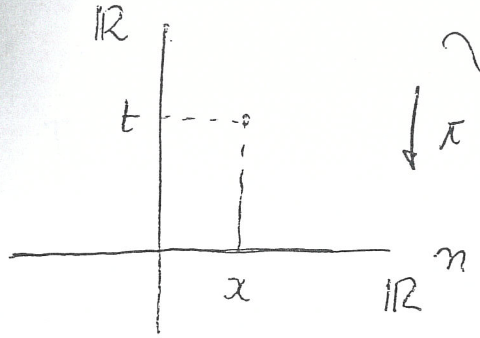
☆☆ The Poincaré lemma (revisited)

de Rham cohomology

ANOTHER APPROACH TO POINCARÉ LEMMA

$$\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \qquad \text{projection}$$
$$(x, t) \mapsto x$$

$$s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R} \qquad \text{null section}$$
$$x \mapsto (x, 0)$$



$$\rightsquigarrow \quad \Omega^*(\mathbb{R}^n \times \mathbb{R}) \xrightarrow{s^*} \Omega^*(\mathbb{R}^n) :$$
$$\Omega^*(\mathbb{R}^n) \xrightarrow{\pi^*} \Omega^*(\mathbb{R}^n \times \mathbb{R})$$

At the cohomological level, s^* e π^* are each other's inverses, as we shall see, so that

$$\boxed{H^*(\mathbb{R}^{n+1}) \cong H^*(\mathbb{R}^n)} \quad ▩$$

whence:

$$\boxed{\text{Poincaré} \; ; \quad H^*(\mathbb{R}^n) = H^*\{pt\} = \begin{cases} \mathbb{R} & \dim = 0 \\ 0 & \text{otherwise} \end{cases}}$$

⚠ consequence

$d\omega = 0 \implies$ locally $\omega = d\alpha$

more generally

$$\boxed{H^*(M \times \mathbb{R}) \cong H^*(M)}$$

(same proof as ▩)

"⇐" holds as well → remember that d is a local operator

Let us prove

From $\pi \circ s = 1$ we get $s^* \circ \pi^* = 1$

however $s \circ \pi \neq 1$!

$(\Rightarrow \pi^* \circ s^* \neq 1)$

Nevertheless we shall construct a homotopy operator

$$K : \Omega^*(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{*-1}(\mathbb{R}^n \times \mathbb{R})$$

(lowering degree by 1) :

such that

$$1 - \pi^* \circ s^* = \pm \underbrace{(dK \pm Kd)}_{\downarrow}$$

closed forms are then mapped to exact forms

$\Rightarrow \quad 1 = \pi^* \circ s^*$ in cohomology.

($\pi^* \circ s^*$ is chain-homotopic to 1)

[This is a purely homological algebraic concept]

$$(1 - \pi^* \circ s^*) \omega = \pm (dK\omega \pm K d\omega)$$

$$= (\text{if } d\omega = 0) = \pm d(K\omega)$$

ω closed $\qquad$ an exact form

Construction of K

Every form on $\mathbb{R}^n \times \mathbb{R}$ is a linear combination (in a unique way) of forms of the following two types:

(I) $\quad (\pi^*\phi)\, f(x,t)$ $\qquad \phi$: a form on the "base" $\mathbb{R}^n$

(II) $\quad (\pi^*\phi)\, f(x,t)\, dt$

In other words : (I) $\quad f(x,t)\, dx_1 \dots dx_q$

(II) $\quad f(x,t)\, dx_1 \dots dx_q \; dt$

Define K as follows

$$\boxed{K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})}$$

(I) : $\quad K\big((I)\big) = 0$

(II) : $\quad (\pi^*\phi)\, f(x,t)\, dt \mapsto (\pi^*\phi) \int_0^t f$

"fibre integration"

(I) $\quad \omega = (\pi^*\phi)\, f(x,t) \qquad \deg \omega = q$

$$(1 - \pi^* s^*)\,\omega = (\pi^*\phi)\, f(x,t) - \pi^*\phi \cdot f(x,0)$$

$$(dK - Kd)\,\omega = -K d\omega = -K\Big(\underbrace{[d\pi^*\phi] f}_{K(\cdot) = 0} + (-1)^q \pi^*\phi \left(\frac{\partial f}{\partial x} dx\right)\Big)$$

$$= (-1)^{q-1} \pi^*\phi \left(\int_0^t \frac{\partial f}{\partial t}\, dt\right) = (f(x,t) - f(x,0))\, \pi^*\phi \times (-1)^{q-1}$$

Thus

$$1 - \pi^* s^* = (-1)^{q-1}(dK - Kd). \qquad (I)$$

$\boxed{(II)}$ If now $\omega = (\pi^*\phi) f \, dt$

$\deg \omega = q$

we get

$$d\omega = (\pi^* d\varphi) f \, dt + (-1)^{q-1}(\pi^*\phi) \frac{\partial f}{\partial x} dx \wedge dt$$

$$(1 - \pi^* s^*)\omega = \omega$$

$$(s^* dt = d(s^* t) = d(0) = 0)$$

$$K d\omega = \underbrace{(\pi^* d\phi)\int_0^t f}_{(*)} + (-1)^{q-1}(\pi^*\phi)\int_0^t \frac{\partial f}{\partial x} dx$$

they cancel out

$$dK\omega = d\left(\pi^*\phi \int_0^t f\right)$$

$$= \underbrace{(\pi^* d\phi)\int_0^t f}_{(*)} + (-1)^{q-1}(\pi^*\phi)\left[dx \int_0^t \frac{\partial f}{\partial x} + f \, dt\right]$$

differentiation with respect to a parameter

$\Rightarrow$

$$dK\omega - Kd\omega = (-1)^{q-1}\omega$$

Therefore:

$$\boxed{1 - \pi^* \circ s^* = (-1)^{q-1}(dK - Kd)}$$

$\square$