HOMOTOPY INVARIANCE OF DE RHAM COHOMOLOGY
COHOMOLOGY WITH COMPACT SUPPORTS

Further consequences

The de Rham cohomology fulfils the following

"homotopy axiom"

cf. Eilenberg-Steenrod

$$\boxed{\text{homotopic maps induce the same map in cohomology}}$$

homotopy (or $[0,1]$..)

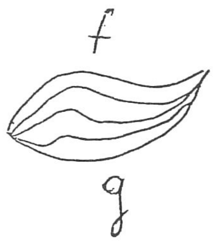
$$F : M \times \mathbb{R} \to N$$

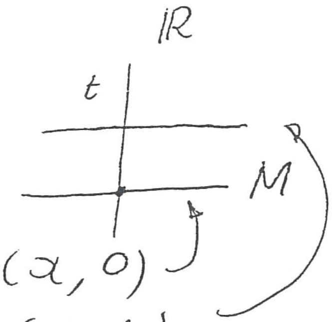
Proof.

$$\begin{tikzcd} M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & N \end{tikzcd}$$

$$F(x,t) = f(x) \qquad t \geq 1$$

$$F(x,t) = g(x) \qquad t \leq 0$$





Set

$$S_{0,1} \quad M \to M \times \mathbb{R}^1$$

$$S_0(x) = (x,0)$$
$$S_1(x) = (x,1)$$

Then

$$f = F \circ S_1$$
$$g = F \circ S_0$$

corresponding maps
$f^*: H^*(N) \to H^*(M)$
are induced ...

Therefore

$$f^* = (F \circ S_1)^* = S_1^* \circ F^*$$
$$g^* = S_0^* \circ F^*$$

However, in cohomology $S_0^* = S_1^* = \pi^{*^{-1}}$

$$\Rightarrow \quad f^* = g^* \quad \text{in cohomology}$$

□

<u>Def.</u> $\star$ M e N have the same <u>homotopy type</u>

(in a C^∞ sense, but this is equivalent to C^0) $\star$)

if $\exists$ $f : M \to N$
$g : N \to M$

such that $f \circ g \underset{\mathcal{C}^\infty\text{- homotopic}}{\sim} 1_N$

$g \circ f \sim 1_M$

$\star$ **M** is said to be <u>contractible</u> whenever it has the homotopy type of a point

$\star$ It is then clear that if M and N have the same homotopy type, then they possess the same de <u>Rham cohomology</u>

Let $i : A \hookrightarrow M$ <u>inclusion</u>

e $r : M \to A$: r <u>retraction</u>.

with $r|_A = id$

$(r \circ i = id)$

If $\quad i \circ r : M \to M$

is homotopic to id_M

we have a <u>deformation</u> <u>retraction</u>

that is

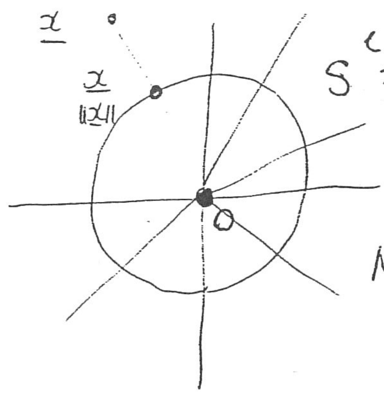
A is a (deformation) <u>retract</u> of M

then A has the same <u>homotopy</u> type

as M, hence $\quad \boxed{H^*(A) \cong H^*(M)}$

Example : $A = S^1 \qquad M = \mathbb{R}^2 \setminus \{\underline{0}\}$

$i =$ inclusion



$S^1 = A$

$r = r(\underline{x}) = \dfrac{\underline{x}}{\|\underline{x}\|} \qquad M = \mathbb{R}^2 \setminus \{\underline{0}\}$

$$i \circ r\,(\underline{x}) = \frac{\underline{x}}{\|\underline{x}\|}$$

is <u>homotopic</u> <u>to the identity</u> $\underline{x} \to \underline{x}$:

$$\boxed{F(\underline{x}, t) = \frac{\underline{x}}{\|\underline{x}\|}\, t + (1-t)\, \underline{x}}$$

$$\Rightarrow H^*(\mathbb{R}^2 - \{0\}) \cong H^*(S^1)$$

The Poincaré lemma for compactly supported cohomology

[same definition as above, but requiring compact supports everywhere]

notice

$$H_c^{*+1}(\mathbb{R}^n \times \mathbb{R}) \cong H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dim } n \\ 0 & \text{otherwise} \end{cases}$$

In general

$$\boxed{H_c^*(M \times \mathbb{R}) \cong H_c^{*-1}(M)}$$

We just outline the main steps of the proofs, emphasising the differences with respect to ordinary cohomology.

⚠ π^* does not send compactly supported forms to compactly supported ones

However, one does have a "push-forward" map

π_* ("fibre integration")

defined as follows

(I) $\underset{\text{not necessarily compactly supported}}{\pi^*\phi} \cdot \underbrace{f(x,t)}_{\text{compact support}} \longmapsto 0$

(II) $\pi^*\phi\, f(x,t)\,dt \longmapsto \phi \int_{-\infty}^{+\infty} f(x,t)\,dt$

This is meaningful!

$$\pi_* : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M)$$

is indeed a <u>chain map</u>:

for the pull-back this was automatic by functoriality

to be shown $\rightsquigarrow$ $\boxed{d\pi_* = \pi_* d}$

and hence it induces a map $\pi_* : H_c^*(M \times \mathbb{R}) \to H_c^{*-1}(M)$

check:

(I) $f(x,t)\,\omega \xmapsto{\pi_*} 0 \xrightarrow{d} 0$

$f(x,t)\,\omega \xmapsto{d} df\,\omega + f\,d\omega$

π_*: $df\,\omega \mapsto 0$ ($\int df = 0$; f has compact support), $f\,d\omega \mapsto 0$

(II) $f(x,t)\,\omega\,dt \xmapsto{\pi_*} \omega \int f(x,t)\,dt$

$\xmapsto{d} d\omega \int f(x,t) + \omega \left(\int \frac{\partial f}{\partial x} dt\right) dx\, dx$

$dx^i \wedge dx^j \ldots$

$f(x,t)\,\omega\,dt \xmapsto{d} df\,\omega\,dt + f\,d\omega\,dt$

$\xmapsto{\pi_*} \int \frac{\partial f}{\partial x} dt\, \omega\, dx\, dx + d\omega \int f(x,t)$

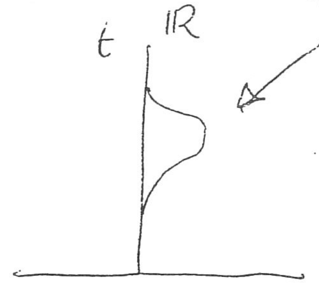
$dx^i \wedge dx^j$

abridged notation

Let us construct a map e_* inverting π_*: it will be the prototype of the Thom class

$$\boxed{\begin{aligned} e_* : \Omega^*_c(M) &\to \Omega^{*+1}_c(M \times \mathbb{R}) \\ \varphi &\mapsto \varphi \wedge e \end{aligned}}$$

$e = e(t)\,dt$



$\int e(t) = 1$

obviously $\quad \pi_* \circ e_* = 1$

but $\quad \boxed{e_* \circ \pi_* \neq 1}$

Again, one builds up a homotopy operator

$$K : \Omega^*_c(M \times \mathbb{R}) \to \Omega^{*-1}_c(M \times \mathbb{R})$$

(I) $K(\phi \cdot f) = 0$

(II) $K(\phi \cdot f\,dt) = \phi \int_{-\infty}^{t} f - \phi \int_{-\infty}^{t} e \int_{-\infty}^{+\infty} f$

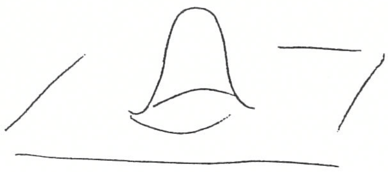
one gets $\quad \boxed{1 - e_* \pi_* = (-1)^{q-1}(dK - Kd)}$

in degree q $\quad$ (see Bott-Tu for details)

Let us produce a generator for $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$

Take 1 for $\{pt\}$ and then iterate e_*: one gets a "bump" n-form

$$e(x_1)e(x_2)\dots e(x_n)\,dx_1 \dots dx_n$$



with "arbitrarily small" support

Obviously H^*_c is not invariant under homotopy equivalences, however it is diffeomorphism invariant (hence a more powerful invariant)

$$\boxed{\text{Corollary: } \mathbb{R}^n \ncong \mathbb{R}^m \text{ if } n \neq m}$$

Theorem of invariance of dimension

Mayer-Vietoris for compactly supported cohomology

Crucial fact

Ω^* : contravariant functor

$f: M \to N \qquad f \in C^\infty(M,N)$

$f^*: \Omega^*(N) \to \Omega^*(M)$

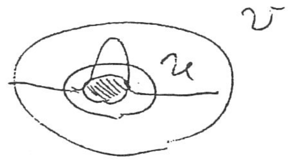
pull-back

allows definition of degree for a proper map

Ω^*_c : the same, but just for proper maps (f^{-1}(compact) = compact.)

covariant functor with respect to inclusion of open sets

$j : U \hookrightarrow V$



$j_* : \Omega^*_c(U) \hookrightarrow \Omega^*_c(V)$

(extends to zero...)

Let us consider

disjoint union

$$\bar{U} \cap \bar{V} \rightrightarrows U \coprod \bar{V} \to M$$

$$\Omega_c^*(\bar{U}\cap\bar{V}) \xrightarrow{\delta} \Omega_c^*(\mathcal{U}) \oplus \Omega_c^*(\bar{V}) \xrightarrow{\text{Sum}} \Omega_c^*(M)$$

$$\omega_1 \qquad \omega_2 \qquad \omega_1 + \omega_2$$

signed inclusion

$$\omega \mapsto (-j_* \omega, j_* \omega)$$

one gets the short exact sequence of Mayer-Vietoris with compact supports

$$\boxed{0 \to \Omega_c^*(\bar{U}\cap\bar{V}) \to \Omega_c^*(\bar{U}) \oplus \Omega_c^*(\bar{V}) \to \Omega_c^*(M) \to 0}$$

Dim : easy.....

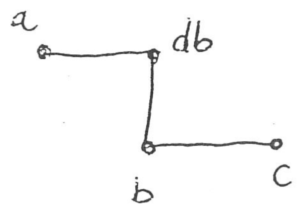
Notice that now

$\rho_{\bar{U}}\, \omega$ is a form on $\mathcal{U}$

whilst in the previous construction it was a form on $\bar{V}$!

Then, "by abstract nonsense"

one obtains a corresponding long exact sequence in cohomology



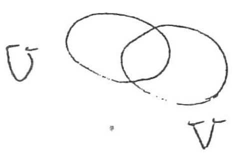
$$\cdots \to H_c^{q}(M) \xrightarrow{d_*} H_c^{q+1}(U \cap V) \to H_c^{q}(U) \oplus H_c^{q}(V) \to H_c^{q}(M) \to \cdots$$

$$\cdots \xrightarrow{d_*} H_c^{q}(U \cap V) \to H_c^{q}(U) \oplus H_c^{q}(V) \to H_c^{q}(M) \xrightarrow{d_*} \cdots$$

Ex: Let us compute $H_c^*(S^1)$ $\left(= H^*(S^1) \, ! \right)$

	$U \cap V$		$U \amalg V$		S^1
0	0		0	$\to$	?
1	$\mathbb{R} \oplus \mathbb{R}$	$\xrightarrow{\delta}$	$\mathbb{R} \oplus \mathbb{R}$	$\to$	?
2	0		0		

$$\delta : \quad \omega = (\omega_1, \omega_2) \longmapsto \left(-(j_U)_* \omega, (j_V)_* \omega \right)$$



dim Im δ = dim Ker $\delta = 1$

$$\boxed{\begin{aligned} H_c^0(S^1) &= \operatorname{Ker} \delta = 1 \\ H_c^1(S^1) &= \operatorname{coker} \delta = 1 \end{aligned}}$$

★ Aside $\quad B^{n-1} \overset{\text{diff}}{\simeq} \mathbb{R}^n \qquad n \geq 2$

$$\| $$

$$\{(x_i) \in \mathbb{R}^n \,/\, \sum_{i=1}^{n} x_i^2 < 1\}$$

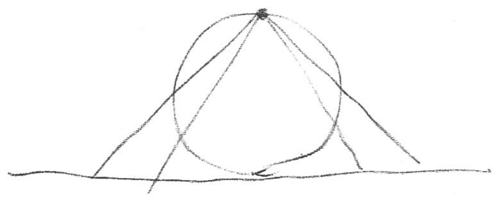
Proof. Firts

$$I = (a,b) \simeq S^1 - \{N\} \simeq \mathbb{R}$$

via stereographic projection

or

$$I \simeq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \underset{tg}{\simeq} \mathbb{R}$$



$0 \leq r < 1$

★ generalised spherical coordinates

Now

$0 \to r = 0$

$$\begin{cases} x_1 = r \cos\theta_1 \ldots \cos\theta_{n-1} \\ x_2 = r \cos\theta_1 \ldots \sin\theta_{n-1} \\ x_3 = r \cos\theta_1 \ldots \cos\theta_{n-2} \\ x_4 = r \cos\theta_1 \ldots \sin\theta_{n-2} \\ \vdots \\ x_n = r \sin\theta_1 \end{cases}$$

$\theta_{n-1} \in [0, 2\pi)$

$\theta_i \in [0, \pi]$

$i = 1, 2, \ldots n-2$

achieving the conclusion

★ One may use cubes as well  $\simeq$  $\simeq \mathbb{R}^n$