THE FIVE LEMMA
POINCARÉ DUALITY

Lecture LIV

★ The "Five lemma"

$$\begin{tikzcd}
\arrow[r] & A \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B \arrow[r, "f_2"] \arrow[d, "\beta"'] & C \arrow[r, "f_3"] \arrow[d, "\gamma"'] & D \arrow[r, "f_4"] \arrow[d, "\delta"'] & E \arrow[r] \arrow[d, "\varepsilon"'] & \dots \\
\arrow[r] & A' \arrow[r, "f_1'"'] & B' \arrow[r, "f_2'"'] & C' \arrow[r, "f_3'"'] & D' \arrow[r, "f_4'"'] & E' \arrow[r] & \dots
\end{tikzcd}$$

- Let $A, A' \dots$ abelian groups ; $\alpha, \dots f, f'$ homomorphisms
- The diagram is commutative with exact rows
- $\alpha, \beta, \delta, \varepsilon$ are isomorphisms

✦ Then γ is an isomorphism

Proof ① γ is injective

Let $\gamma(c) = 0 \quad (c \in C)$

Then $\delta f_3(c) = f_3'(\gamma(c)) = 0$

$\Rightarrow \quad \boxed{f_3(c) = 0}$ ★

$\Rightarrow \quad c \in \operatorname{Im} f_2 : \exists\, b \in B$ with

$\boxed{c = f_2(b)}$

and $\gamma f_2(b) = 0 \quad \Rightarrow \quad \boxed{f_2' \beta(b) = 0}$

$(\beta(b) \in \operatorname{Ker} f_2')$. But $\operatorname{Ker} f_2' = \operatorname{Im} f_1'$

$\Rightarrow \exists\, a' \in A'$ s.t.

$$\boxed{f_1'(a') = \beta(b)}$$

and $a' = \alpha(a)$ <u>for a unique</u> a

Therefore

$$f_1'(\alpha(a)) = \beta(b)$$

$$\Rightarrow \beta(f_1(a)) = \beta(b)$$

$$\Rightarrow \boxed{b = f_1(a)}$$

and $c = f_2(b) = f_2(f_1(a)) = 0$

$\square$

(2) <u>γ is surjective</u>

Let $c' \in C'$

$f_3'(c') \in \operatorname{Ker} f_4'$: $f_4'(f_3'(c')) = 0$

$\Rightarrow \varepsilon^{-1}[f_4'(f_3'(c'))] = 0$

$\Rightarrow \boxed{f_4[\delta^{-1} f_3'(c')] = 0}$

$$\Rightarrow \quad \delta^{-1} f_3'(c') = f_3(c) \quad \text{for some } c \in C$$

$$\Rightarrow \quad f_3'(c') = \delta f_3(c) = f_3' \gamma(c)$$

Hence $\forall\, c', \exists\, c$ such that

$$\boxed{f_3'(c' - \gamma(c)) = 0}$$

Subsequently we have

$$c' - \gamma(c) = f_2'(b') \quad \text{for some } b'$$

$$\| \qquad \text{[commutativity of the diagram]}$$

$$\gamma f_2(\beta^{-1} b')$$

$$= \quad c' = \gamma(\underbrace{c + f_2(\beta^{-1}(b'))}_{\equiv\, \tilde{c}})$$

$$\Rightarrow \quad c' = \gamma(\tilde{c}) \qquad \square$$

Poincaré duality

If M is orientable (and without boundary), there exists a **pairing**:

$$H^q(M) \times H_c^{n-q}(M) \to \mathbb{R}$$

$$\left[\int : \quad H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R} \right]$$

induced by $\boxed{\int \omega \wedge \tau}$

($\int$ is well-defined : • d is an antiderivation
see next page for details
• Stokes ← M is orientable

Theorem: If M is orientable and possesses a finite good covering

the pairing $\int$ is dual — that is

$$\begin{cases} \langle v, w \rangle = 0 \ \forall w \Rightarrow v = 0 \\ \langle v, w \rangle = 0 \ \forall v \Rightarrow w = 0 \end{cases}$$

$\left(\langle , \rangle \; V \otimes_{\mathbb{R}} W \to \mathbb{R} \right.$

$\left. \Rightarrow \quad \boxed{V \cong W^*} \right)$

hence

$$\boxed{H^q(M) \cong \left[H_c^{n-q}(M) \right]^*}$$

more general validity

(one also has $V^* \cong W$)

We check that $\int$ goes down to cohomology:
that is, it is well-defined

$d\omega = 0$
$d\tau = 0$

$$\int_M (\omega + da) \wedge (\tau + \underbrace{db}_{\text{supp. compatto}}) =$$

$$= \int_M \omega \wedge \tau + \int_M da \wedge \tau + \int_M \omega \wedge db + \int_M da \wedge db$$

$$= \int_M \omega \wedge \tau \pm \underbrace{\int_M d(a \wedge \tau)}_{\int_{\partial M} a \wedge \tau = 0} \pm \int_M a \wedge \underbrace{d\tau}_{0}$$

$$\pm \underbrace{\int_M d(\omega \wedge db)}_{0} \pm \int_M \underbrace{d\omega}_{0} \wedge b$$

$$\pm \underbrace{\int_M d(a \wedge db)}_{0} = \int_M \omega \wedge \tau$$

Comment: One has to show that

$$(*) \quad \int_M \omega \wedge \tau = 0 \quad \forall \tau,\ d\tau = 0, \quad d\omega = 0 \quad \Rightarrow \quad \omega = d\alpha$$

and conversely.

This is not straightforward, although

$$\int_M \omega \wedge \tau = 0 \quad \forall \tau \quad \Rightarrow \quad \omega = 0$$

Aside: Another proof, for compact, boundary-free manifolds can be obtained via Hodge theory, which requires a riemannian metric.

We shall use

$$\boxed{\text{Mayer-Vietoris} + \text{five lemma}}$$

<u>Proof</u> The two Mayer-Vietoris sequences can be <u>paired</u> and give rise to a <u>commutative</u> <u>diagram</u> "up to signs"

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(\bar U \cup \bar V) \arrow[r, "\text{restriction}"] \arrow[d, dashed] \arrow[d, phantom, shift right=4ex, "\otimes"] & H^q(\bar U)\oplus H^q(\bar V) \arrow[r, "\text{difference}"] \arrow[d, dashed] \arrow[d, phantom, shift right=4ex, "\otimes"] & H^q(\bar U \cap \bar V) \arrow[r, "d^*"] \arrow[d, dashed] & H^{q+1}(\bar U \cup \bar V) \\
\cdots & H_c^{n-q}(\bar U \cup \bar V) \arrow[l] \arrow[d, "\int_{\bar U \cup \bar V}"] & H_c^{n-q}(\bar U)\oplus H_c^{n-q}(\bar V) \arrow[l, "\text{sum}"'] \arrow[d, "\int_{\bar U} + \int_{\bar V}"] & H_c^{n-q}(\bar U \cap \bar V) \arrow[l, "\text{signed inclusion}"'] \arrow[d, "\int_{\bar U \cap \bar V}"] & \cdots \arrow[l, "d_*"'] \\
& \mathbb{R} & \mathbb{R} & \mathbb{R} &
\end{tikzcd}$$

$$\int_{\bar U \cap \bar V} \omega \wedge d_* \tau = \pm \int_{\bar U \cap \bar V} d^* \omega \wedge \tau$$

et cetera

So two "squares" are commutative. Let us check the last one

Vediamo l'ultimo

$$\begin{tikzcd}
H^q(U\cap V) \arrow[r, "d^*"] & H^{q+1}(U \cup V) \\
\otimes & \otimes \\
H_c^{n-q}(U\cap V) & H_c^{n-(q+1)}(U\cap V) \arrow[l, "d^*"]
\end{tikzcd}$$

$$\left[\, d^* w = \begin{cases} -d(\rho_U w) & \text{on } U \\ d(\rho_U w) & \text{on } V \end{cases} \right.$$

$$(\in H^{q+1}(U \cup V))$$

$$\left[\, d_* \tau \to (d\rho_U \tau,\ d(\rho_V \tau)) \right.$$

$$ \| \qquad\qquad \|$$

$$ d\rho_U \tau \qquad d\rho_V \tau \qquad (\text{why?}) \,\Big]$$

$$\boxed{\int_{U\cap V} \omega \wedge d_* \tau} = \int_{U \cap V} \omega \wedge d\rho_V \tau$$

$$= \underbrace{(-1)^{\deg w} \int_{U\cap V} d\rho_U \omega \wedge \tau}_{= \pm}$$

On the other hand

$$\int_{U\cup V} \underbrace{d^* w}_{\text{has support in}} \wedge \tau = \overbrace{-\int_{U\cap V} d\rho_U w \wedge \tau}.$$

If Poincaré duality holds for U, V, $U \cap V$, then it holds for $U \cup V$ (☆ five lemma)

☆ It certainly holds for $\mathbb{R}^n$ (Poincaré lemmata)

$\Rightarrow$ (induction) it holds for M having a finite good covering

$\square$

Let us summarise the layout of the proof in a slitly different manner

(One first checks that

$$\begin{array}{ccccccccc}
\to & H^q & \to & H^q \oplus H^q & \to & H^q & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & (H_c^{n-q})^* & \to & (H_c^{n-q})^* \oplus (H_c^{n-q})^* & \to & (H_c^{n-q})^* & \to
\end{array}$$

is commutative up to signs

Then one uses the five lemma

$$\begin{array}{ccccccccc}
0 & \to & 0 & \to & H^q(U \cup V) & \to & 0 & \to & 0 \\
\downarrow \text{iso} & & \downarrow \text{iso} & & \downarrow \text{iso} & & \text{iso} \downarrow & & \text{iso} \downarrow \\
0 & \to & 0 & \to & H_c^{n-q}(U \cup V) & \to & 0 & \to & 0
\end{array}$$

One may drop the assumption on the good covering

(c.v. Greub, Halperin, Van Stone,

o De Marco). Si ha

$$\boxed{H^q(M) \simeq \left[H_c^{n-q}(M)\right]^* \quad \forall q}$$

The statement is asymmetric !

Indeed, if $M = \coprod_{i=1}^{\infty} M_i$ disjoint union

finite dimensional cohomology

$$H^q(M) = \prod_i H^q(M_i) \qquad \text{(direct product)}$$

$$H^q_c(M) = \bigoplus_i H^q_c(M_i) \qquad \text{(direct sum)}$$

(involves finite sums.)

$$\boxed{\begin{array}{l} \text{Now} \quad (\bigoplus_i V_i)^* = \prod_i V_i^* \\ \text{But} \quad (\prod_i V_i)^* \neq \bigoplus_i V_i^* \ ! \end{array}}$$

(otherwise $V \cong V^{**}$

which is false in infinite dimensions)

algebraic bidual

In our case ◆ holds if Poincaré holds for M_i.

Consequences

(1) M oriented, connected, $\dim M = n$, $\partial M = \phi$

$\Rightarrow \boxed{H^n_c(M) = \mathbb{R}}$ ($\Rightarrow$ se M è compact, $\boxed{H^n(M) = \mathbb{R}}$

Indeed $H^0(M) = \mathbb{R} = H^n_c(M)^*$ □

(2) M as before, non compact

$\Rightarrow \boxed{H^n(M) = 0}$

Pf: $H^n(M) \cong H^0_c(M)^* = \dots = 0$
why?