⚡ geometric Poincaré Duality (hint)

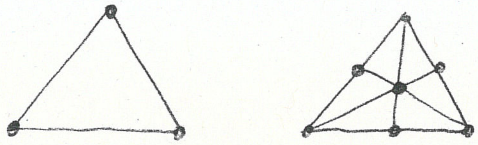
for simplicial complexes K

Lecture LIX

POINCARÉ DUALITY AGAIN

Let us consider $K^1 \equiv$ first barycentric subdivision

(ex:



" dual decomposition "

σ^0_α vertex in K

$$*\sigma^0_\alpha = \bigcup \tau^n_\beta \qquad n\text{-cell}$$

$$\tau^n_\beta \ni \sigma^0_\alpha$$

n-simplices $\ni \sigma^0_\alpha$

in general

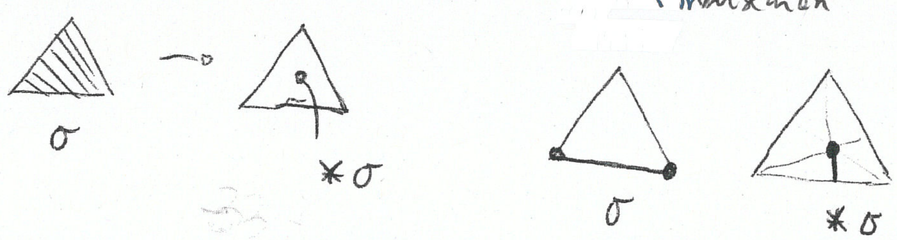
$$\left[\; *\sigma^k_\alpha := \bigcap_{\sigma^0_\beta \in \sigma^k_\alpha} *\sigma^0_\beta \;\right]$$

$\shortparallel$

Δ^{n-k}_α the only cell intersecting σ^k_α transversally. The orientation is chosen so as to have $i_p(\sigma, \Delta) = +1$

↖ intersection

ex:



Cap , Cup , wedge

intersection $\cup$ $\wedge$

$\cap$

(algebra structures in cohomology (singular & de Rham) and homology (singular)

M compact oriented manifold $\dim M = n$

$$\left[\begin{array}{ll} \sigma & k\text{-cycle} \\ \tau & n-k \text{ cycle} \end{array}\right]$$

Poincaré duals (another version)

φ_σ , φ_τ

we require $$\int_\mu \varphi_\sigma = \#(\sigma \cap \mu) \equiv \sigma \cdot \mu$$ μ $(n-k)$-cycle

$$\int_\nu \varphi_\tau = \tau \cdot \nu$$ ν k-cycle

Existence guaranteed: de Rham

Remark: if σ or τ is torsion , $\varphi_\sigma \, (\varphi_\tau) \equiv 0$

since $\sigma \cdot \mu = 0 \quad \forall \mu \ldots$

in particular: $$\int_M \varphi_{\sigma \cap \tau} = \sigma \cdot \tau$$

Let

$$\begin{array}{ccc} M \times M & \xrightarrow{\pi_2} & M \\ {\scriptstyle \pi_1}\downarrow & & \\ M & & \end{array}$$

One has:

$$\int_{\mu \times \nu} \pi_1^* \varphi_\sigma \wedge \pi_2^* \varphi_\tau = \int_\mu \varphi_\sigma \int_\nu \varphi_\tau = (\sigma \cdot \mu)(\tau \cdot \nu)$$

Let $(p_1, p_2) \in \sigma \times \tau \cap \mu \times \nu$

$$i_{(p_1,p_2)}(\sigma \times \tau, \mu \times \nu) = (-1)^{n-k} i_{p_1}(\sigma \cdot \mu)\, i_{p_2}(\tau \cdot \nu)$$

$$\Rightarrow \boxed{(\sigma \times \tau \cdot \mu \times \nu) = (-1)^{n-k} \int_{\mu \times \nu} \pi_1^* \varphi_\sigma \wedge \pi_2^* \varphi_\tau}$$

holds $\forall$ $n-k'$ cycle μ, k cycle ν — if $k \neq k'$ one has $0 = 0$

However (**Künneth** for $H_*(M, \mathbb{R})$)

such cycles $\mu \times \nu$ generate $H_n(M \times M, \mathbb{R})$

$\Rightarrow$ $\pi_1^* \varphi_\sigma \wedge \pi_2^* \varphi_\tau$ = Poincaré dual

of $(-1)^{n-k}\, \sigma \times \tau$ namely

$$(-1)^{n-k}\int_{\eta}\underbrace{\pi_1^*\varphi_\sigma\wedge\pi_2^*\varphi_\tau}_{}\ \ \ \ = \ \ \sigma\times\tau\cdot\eta$$

w

η: n-cycle in $M\times M$

Now let $\eta = \Delta \subset M\times M$ <u>diagonal</u>

One has: $$\int_\Delta w = \int_\Delta \varphi_\sigma\wedge\varphi_\tau$$

now $(p,p)\in\sigma\times\tau\cap\Delta$ corresponds to $p\in\sigma\cap\tau$ important

and
$$i_{(p,p)}(\sigma\times\tau\cap\Delta) = (-1)^{n-k}\, i_p(\sigma\cap\tau)$$

$$\Rightarrow \boxed{\sigma\cdot\tau = (-1)^{n-k}(\sigma\times\tau\cdot\Delta)}$$

$$\int_M \varphi_{\sigma\cap\tau} = \int_M \varphi_\sigma\wedge\varphi_\tau$$

$$\Rightarrow \boxed{\varphi_{\sigma\cap\tau} = \varphi_\sigma\wedge\varphi_\tau}$$ (as classes)

The formula, proven for $k=(n-k)$, holds in general

Let us retrieve the "old" Poincaré dual

First $\left[\varphi_{f^{-1}(\sigma)} = f^{*}\varphi_{\sigma} \right]$

naturality with respect to pull-back & preimage

$$H^{k}_{DR}(M) \otimes H^{n-k}_{DR}(M) \longrightarrow \mathbb{R}$$

$$([\varphi];[\psi]) \longmapsto \int_M \varphi \wedge \psi$$

the pairing is dual, as we already know

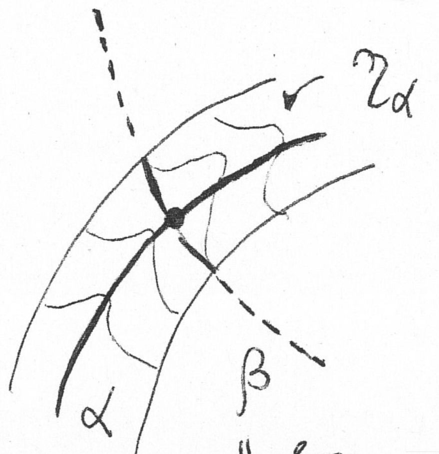
$\Rightarrow$ $\forall$ k-form ψ, $d\psi = 0$, $\exists!$ (in homology) cycle A: $\int_M \varphi \wedge \psi = \int_A \psi$

$$\boxed{\varphi = \varphi_A \equiv \eta_A}$$

old notation

Remark (semi heuristic, but correct)

M



β || loc: fibre of a tubular neighborhood of α

$$\overset{k}{\alpha} \cdot \overset{n-k}{\beta} = \int_M \eta_\alpha \wedge \eta_\beta$$

$$= \int_\beta \eta_\alpha \qquad \Big(\int_{\text{fibra}} \eta_\alpha = 1 \ldots \Big)$$

"cup"

$$\left[\underset{H^k(M,\mathbb{R})}{\alpha} \cup \underset{H^{k'}(M,\mathbb{R})}{\beta} := \Delta^*(\alpha \otimes \beta)\right]$$

($\Delta : M \to M \times M$ diagonal)

singular cohomology

with $\underset{H^{k+k'}(M \times M)}{\alpha \otimes \beta} \ (\sigma \times \tau) = \alpha(\sigma)\, \beta(\tau)$

$\alpha \mapsto \alpha_{DR}$

$\beta \mapsto \beta_{DR}$

$\pi_1^* \alpha_{DR} \wedge \pi_2^* \beta_{DR}$ represents $\alpha \otimes \beta$ (Künneth)

$\Rightarrow \quad \cup \longleftrightarrow \wedge$

de Rham

$\Rightarrow$ the de Rham isomorphism is an <u>algebra</u> isomorphism