THE KÜNNETH FORMULA

☆ The Künneth formula

manifolds

$$\begin{array}{ccc} M \times F & \xrightarrow{p} & F \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

The map $\omega \otimes \phi \longmapsto \pi^* \omega \wedge p^* \phi$

induces a map

$$\boxed{\Psi : H^*(M) \otimes H^*(F) \longrightarrow H^*(M \times F)}$$

check: $da \otimes \phi \longmapsto \pi^* da \wedge p^* \phi$

$(d\phi = 0)$ $= d\pi^* a \wedge p^* \phi$

$= d(\pi^* a \wedge p^* \phi)$ ← exact

Similarly $\qquad (\, d p^* \phi = p^* d\phi = 0 .. \,)$

$\omega \otimes db \longmapsto$ exact form

Theorem. (Künneth): Ψ is an isomorphism

$$H^m(M \times F) \cong \bigoplus_{k=0}^{m} H^{m-k}(M) \otimes H^k(F)$$ ☆

☆ Remark: Künneth is a purely homological algebraic result. Essentially, it asserts that

$$H^*(C_1 \otimes C_2) \cong H^*(C_1) \otimes H^*(C_2)$$

differential complexes

See MacLane "Homology"
◦ Dubrovine Fomenko Novikov

Beware! For $\mathbb{Z}$-modules the statement is more involved

First $\boxed{H^*(\mathbb{R}^m \times F) = H^*(F)}$ ⚡ Poincaré

Proof (hint)

One <u>tensorises</u> M.V. with $H^{n-p}(F)$ (at level p), and one still has an <u>exact sequence</u> [⚠ This is false in general for A-modules]

one has an <u>exact sequence</u>

$$\cdots \to \bigoplus_{p=0}^{n} H^p(\bar U \cup \bar V) \otimes H^{n-p}(F) \quad (A)$$

$$\to \bigoplus_{p=0}^{n} \left(H^p(\bar U) \otimes H^{n-p}(F)\right) \oplus \left(H^p(\bar V) \otimes H^{n-p}(F)\right) \quad (B)$$

$$\to \bigoplus_{p=0}^{n} H^p(\bar U \cap \bar V) \otimes H^{n-p}(F) \to \cdots \quad (C)$$

(see LV-4) ⚡

and a <u>commutative diagram</u>

$$\begin{array}{ccccccc}
A & \to & B & \to & C & \xrightarrow{d^*} & \cdots \\
\downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \\
H^n((\bar U \cup \bar V) \times F) & \to & H^n(U \times F) \oplus H^n(V \times F) & \to & H^n(\bar U \cap \bar V) \times F & \xrightarrow{d^*} & \cdots
\end{array}$$

<u>let us check this</u>

$$
\begin{tikzcd}
\bigoplus_{p=0}^{n} \left(H^{p}(U \cap V) \otimes H^{n-p}(F) \right) \arrow[r, "d^*_M \otimes i_{n+1}"] \arrow[d, "\psi"] & \bigoplus_{k=0}^{n+1} \left(H^{k}(U \cup V) \otimes H^{n+1-k}(F) \right) \arrow[d, "\psi"] \\
H^{n}(\underbrace{(U \cap V) \times F}_{(U \times F) \cap (V \times F)}) \arrow[r, "d^*_{M \times F}"] & H^{n+1}(\underbrace{(U \cup V) \times F}_{(U \times F) \cup (V \times F)})
\end{tikzcd}
\qquad (*)
$$

see next page

Let

$$\omega \otimes \phi \in H^{p}(U \cap V) \otimes H^{n-p}(F)$$

$$\boxed{\psi \, \overset{d^*_M \otimes i}{\overset{\equiv}{d^*}} (\omega \otimes \phi) = \pi^*(d^* \omega) \wedge \rho^* \phi}$$

(the same)

$$\boxed{\overset{d^*_{M \times F}}{\overset{\shortparallel}{d^*}} \psi (\omega \otimes \phi) = d^*(\pi^* \omega \wedge \rho^* \phi)}$$

$$= d\left((\pi^* \rho_U) \pi^* \omega \wedge \rho^* \varphi \right) \quad \text{on } V$$

$$= d \pi^* (\rho_U \omega) \wedge \rho^* \varphi \qquad (d\varphi = 0)$$

$$= \pi^* d(\rho_U \omega) \wedge \rho^* \varphi$$

$$= \boxed{\pi^* d^* \omega \wedge \rho^* \varphi}$$

$$= \boxed{\psi d^* (\omega \otimes \phi)}$$

similar reasoning on U

usage of
- * Poincaré lemma,
- * good covering
- * five lemma

yields the conclusion $\Rightarrow$ The theorem holds for

$\underline{\underline{H_c}}$ as well

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(*) Attention...

$$\underset{H^p}{\omega} \otimes \underset{H^{n-p}}{\phi} \longmapsto \underset{\substack{\cap \\ H^{p+1}}}{d_M^* \omega} \otimes \underset{\substack{\cap \\ H^{n-p}}}{\phi} \longmapsto .. 0$$

Set , $p+1 = k$

Then

$$n - p = n+1 - (p+1)$$
$$= n+1 - k$$

check: the sequence

$$\bigoplus_{p=0}^{n} H^p(U \cap V) \otimes H^{n-p}(F) \xrightarrow{d_M^* \otimes i} \boxed{\bigoplus_{k=0}^{n+1} H^k(U \cup V) \otimes H^{n+1-k}(F)}$$

$$\longrightarrow \bigoplus_{k=0}^{n+1} \left[H^k(U) \otimes H^{n+1-k}(F)\right] \oplus \left[H^k(V) \otimes H^{1+n-k}(F)\right]$$

is exact , the $\boxed{}$

* Variant. From the short exact sequence of complexes (M.V.)

$$0 \to \Omega^*(\mathcal{U} \cup \mathcal{V}) \to \Omega^*(\mathcal{U}) \oplus \Omega^*(\mathcal{U}) \to \Omega^*(\mathcal{U} \cap \mathcal{V}) \to 0$$

one has, upon tensorisation with $\Omega^*(F)$

$$0 \to \Omega^*(\mathcal{U} \cup \mathcal{V}) \otimes \Omega^*(F) \to \begin{array}{c} (\Omega^*(\mathcal{U}) \otimes \Omega^*(F)) \\ \oplus \\ \Omega^*(\mathcal{V}) \otimes \Omega^*(F) \end{array} \to \Omega^*(\mathcal{U} \cap \mathcal{V}) \otimes \Omega^*(F) \to 0$$

which becomes long exact in cohomology

$$\left(D(\omega \otimes \phi) := d_M \omega \otimes \phi \pm \omega \otimes d_F \phi \right)$$

differential of $\bullet \otimes \bullet$

$$D = d_M \otimes 1 \pm 1 \otimes d_F \quad (\dots D^2 = 0)$$

One then proceeds as before.

Digression (quite short) on tensorisation vs exactness

Let M', M, M'', N A-modules

Then

① $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ exact

$\Rightarrow$ $M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0$

$(\otimes = \otimes_A)$ exact

however

② if the last arrow "$\to 0$" is missing

the implication is false in general

(If N is a vector space (A = K field) then it is ok)

Example (see Atiyah - Macdonald p. 29)

$A = \mathbb{Z}$

$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ $\quad f(x) = 2x$ $\quad \boxed{f \text{ is injective}}$

$N = \mathbb{Z}/_2$

$0 \to \mathbb{Z} \otimes \mathbb{Z}/_2 \xrightarrow{f \otimes 1} \mathbb{Z} \otimes \mathbb{Z}/_2$? $\quad f \otimes 1$ injective?

$(f \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes \overset{0}{\overset{\shortparallel}{2y}} = 0$!

$\Rightarrow f \otimes 1 \equiv 0$ ma $\mathbb{Z} \otimes \mathbb{Z}/_2 \neq 0$!

NO

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use ... Künneth formula to verify the table below

yielding $H^*(\Sigma_{g_1} \times \Sigma_{g_2})$

↑ Riemann surfaces

0	1	2	3	4
$\mathbb{R}$	$\mathbb{R}^{2(g_1+g_2)}$	$\mathbb{R}^{2+8g_1g_2}$	$\mathbb{R}^{2(g_1+g_2)}$	$\mathbb{R}$

compute $H^*(S^3 \times S^1)$