Poincaré dual of a closed oriented submanifold

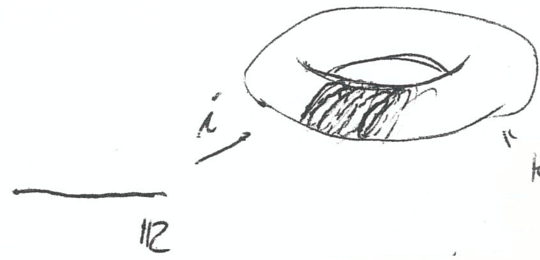
POINCARÉ DUALS

M : smooth, oriented (without boundary) manifold, $\dim M = n$

S : closed oriented submanifold, $\dim = k$

as a subset of M

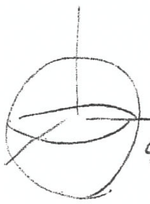
$i : S \hookrightarrow M$



"Kronecker's foliation"

$\mathbb{R}^3 = M$

$i(\mathbb{R})$ not closed



$S^2 = S$ closed

Let $\omega \in \Omega^k_c(M)$

$\mathrm{supp}(\omega|_S)$ is closed in S and also in M

$\cap$

$\mathrm{supp}\, \omega \cap S$, compact

$\Rightarrow$ $\mathrm{supp}(\omega|_S)$ compact

$\Rightarrow$ $i^*\omega$ has compact support

$\Rightarrow$ $\int_S i^*\omega$ is defined.

Now (Stokes) $\int_{\mathcal{S}}$ induces a

<u>linear functional</u> on $H_c^k(M)$, namely

an element of $H_c^k(M)^* \cong H^{n-k}(M)$

(Poincaré)

that is, $\exists ! \, [\eta_{\mathcal{S}}] \in H^{n-k}(M)$

such that

$$\boxed{\int_S i^* \omega = \int_M \omega \wedge \eta_S}$$

$$\boxed{\forall \; \omega \in H_c^k(M)}$$

abuse of notation

$$\boxed{[\eta_{\mathcal{S}}] \equiv \underline{\text{Poincaré}} \; \underline{\text{dual of } \mathcal{S}}}$$

$(\text{o } \eta_{\mathcal{S}} \ldots)$

(<u>closed</u> Poincaré dual)

If S is <u>compact</u>, and if M

has a <u>finite "good covering"</u>

(so one also has $H_c^{k}(M) \cong \left[H^{n-k}(M)\right]^*$)

$\int_S i^*\omega$ is defined $\forall$ <u>p-forms</u>

$\dim S = p$ and one has

$$\int_S i^*\omega = \int_M \overset{p}{\omega} \wedge \overset{n-p}{\eta'_S}$$

$\equiv$ (compact Poincaré dual)

$$[\eta'_S] \in H_c^{n-p}(M)$$

Nf one can take of course $\eta'_S = \eta_S$

but their cohomological classes <u>sono different</u> !!

Ex: $M = \mathbb{R}^n$ $\quad S = P$ <u>point</u> (compact)

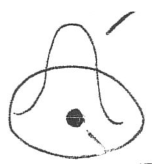
$H^n(\mathbb{R}^n) = 0$

$H^n_c(\mathbb{R}^n) = \mathbb{R}$

$\eta_P = \eta'_P =$ "bump" n-forms concentrated around P supp. arbitrario (... compatto)

(si ... può...)

$\eta_P = \eta'_P$

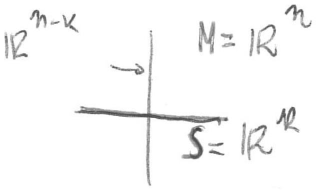


P

$[\eta_P] = 0$ in H^n

$[\eta'_P] \neq 0$ in H^n_c

Further Examples of Poincaré duals

① 

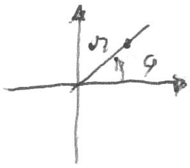
$(n-k)$-form

$$\eta_S = dx^{k+1} \wedge \ldots dx^n$$

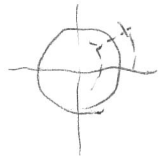
(also $\eta_S = e(x^{k+1}, \ldots, x^n)\, dx^{k+1} \wedge \ldots dx^n$

$\int_{\mathbb{R}^{n-k}} e = 1$

② in $M = \mathbb{R}^2 \setminus \{(0,0)\}$



polar coordinates

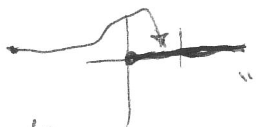
$S^{(1)} = S^1$ 

$[\eta_S] \in H^1(M) \cong H^1(S^1)$ $\quad \eta_S = \rho(r)\,dr$

$\not\parallel$ $\quad\quad \mathbin{\rotatebox{90}{$\cong$}} \quad \cong H^1_c(M)$ $\quad$ $(2-1=1)$ $\quad \int \rho = 1$

$0 \quad\quad \mathbb{R}$ $\quad$ generator

③ in $M = \mathbb{R}^2 \setminus \{(0,0)\}$

$S^{(2)} =$  $= \{(x,0) \,/\, x > 0\}$

$\eta_S = \dfrac{d\varphi}{2\pi}$ $\quad [\eta_S] \in H^1(M) \cong H^1(S^1) \cong \mathbb{R}$

$\quad\quad \cong H^1_c(M)$

Notice:

$$\eta_{S^{(1)}} \wedge \eta_{S^{(2)}} = \rho(r)\, dr \wedge \frac{d\varphi}{2\pi}$$

$S^{(1)} \cap S^{(2)} = \{pt\}$ $\quad \eta_P = \rho(r)\, dr \wedge \dfrac{d\varphi}{2\pi}$ $\quad \int_M \eta_P = 1$

appropriately oriented... 

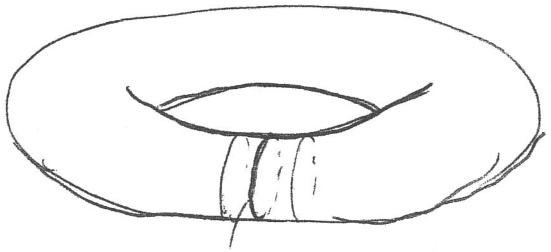
$$\left(\eta_{S^{(1)} \cap S^{(2)}} = \eta_{S^{(1)}} \wedge \eta_{S^{(2)}} \right)$$

see next page

Further examples

(... interpreted as Thom classes)

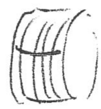
Thom class of the normal bundle to S $\equiv$ Poincaré dual



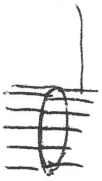
$S \simeq S^1$ $\quad \eta_S = \eta'_S$ $\quad$ (here $H^* = H^*_c$ ...)

$\quad\quad \Vert$

bump 1-form with support in



$$\int_{\text{any fibre}} \eta_S = 1$$

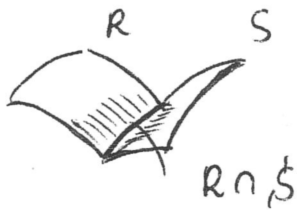


arbitrary
even the full torus!

ex: longitudinal angular form
☆ see below for details.

$\eta_M = 1$ $\quad$ 0-form

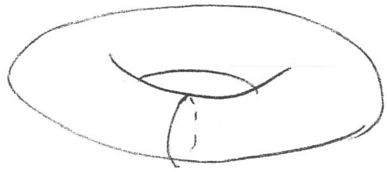
☆ In general (if R and S intersect transversally
(codim $R \cap S$ = codim R + codim S)



$$\boxed{\eta_{R\cap S} = \eta_R \wedge \eta_S}$$ oriented...

$$\boxed{\eta_{\partial S} = d\eta_S \ \ldots}$$ e

Some extra details for tori



S

let us check that

$$\left[\int_{S} i^* w = \int_{T^2} w \wedge \eta_S\right]$$

$\eta_S = \rho(\psi)\, d\psi$

$\int \rho = 1$

$T^2 = S^1 \times S^1$
$\qquad\ \ \, \| $
$\qquad\ \ \, S$
(φ, ψ)

$\forall\ \omega = a(\varphi,\psi)\, d\varphi + b(\varphi,\psi)\, d\psi$

$dw = 0 \qquad \frac{\partial b}{\partial \varphi} - \frac{\partial a}{\partial \psi} = 0$

One can alter w : $w \to w + df \equiv w_\varepsilon$

so as to make w_ε $S^1 \times S^1$ - invariant

$\Rightarrow \omega_\varepsilon = a\, d\varphi + b\, d\psi \qquad a, b \in \mathbb{R}$

(w_ε is the harmonic representative of $[w]$ yielded by Hodge theory

$$\Rightarrow \boxed{\int_{S} i^* w} = \int_{S^1} a\, d\varphi = \int_{S^1} a\, d\varphi \int_{S^1} \rho(\psi)\, d\psi$$

$$= \int_{T^2} a\, d\varphi \wedge \rho(\psi)\, d\psi$$

$$= \int_{T^2} w \wedge \eta_S \Bigg]$$

$\Delta w = 0 \Rightarrow$
$\Delta a = \Delta b = 0$
$\Rightarrow a = \text{cost}$
$\quad\ b = \text{cost}$

Details on Hodge computations

$$\Delta a = 0 \qquad a = a(\varphi, \psi) \qquad a \in \mathcal{C}^\infty(S^1 \times S^1)$$

this is not restrictive by virtue of elliptic regularity

$$\Rightarrow \quad a = \text{cost}$$

One can show this via (double) Fourier series

$$a = \sum_{m,n \in \mathbb{Z}} a_{mn} e^{im\varphi + in\psi} \qquad \begin{array}{l} \varphi \in [0, 2\pi] \\ \psi \in [0, 2\pi] \end{array}$$

$$\frac{d}{d\varphi} \to im \times \cdot$$

$$\Rightarrow \quad \Delta a = 0 \quad \Rightarrow \quad (m^2 + n^2) a_{mn} = 0 \quad \forall (m,n) \in \mathbb{Z}^2$$

$$\Rightarrow \quad a_{mn} = 0 \quad (m, n) \neq (0,0) \qquad \square$$

Aside: it is not difficult to show that

$$f \in \mathcal{C}^\infty(\mathbb{T}^2) \iff \underbrace{\{f_{m,n}\}}_{\text{Fourier coefficients}} \in \underbrace{\mathcal{S}(\mathbb{Z}^2)}_{\text{Schwartz sequences}},$$

that is

$$\lim_{|m|+|n| \to +\infty} \overset{\text{polynomial}}{P(m,n)} f_{mn} = 0$$

$\forall$ P polynomial.

★ Aside

If V is an analytic submanifold of $\mathbb{P}^n$, $\dim_{\mathbb{C}} V = d$.

then

$$\mathrm{vol}(V) = \frac{1}{d!}\int_V \omega_{FS}^d > 0$$

★ ω_{FS} : kähler 2-form (Fubini-Study)

(Special case of Wirtinger's theorem)

Fact:

$$\Rightarrow \boxed{[\omega_{FS}] \in H^2(\mathbb{P}^n, \mathbb{R})}$$

★ generates the cohomology $H^*(\mathbb{P}^n, \mathbb{R})$

(ex: $n=1$ $\mathbb{P}^1 \cong S^2$ stereographic projection)

ω_{FS} = area form

[ω_{FS} cannot be exact

$$\omega^k = d\psi \Rightarrow \underbrace{\int_{\mathbb{P}^n} \frac{\omega^n}{n!}}_{>0} = .. \int_{\mathbb{P}^n} d(\psi \wedge \omega^{n-k}) = 0$$

(Stokes)]

Poincaré Duality ★

$$1 \longleftrightarrow \mathbb{P}^n$$

$$\omega_{FS} \longleftrightarrow \mathbb{P}^{n-2}$$

$$\omega_{FS}^k \longleftrightarrow \mathbb{P}^{n-k}$$

$$\mathbb{P}^{n-k_1} \overset{\text{intersection}}{\cdot} \mathbb{P}^{n-k_2} = \pm \mathbb{P}^{n-k_1-k_2}$$

$$\omega_{FS}^{k_1} \wedge \omega_{FS}^{k_2} = \omega_{FS}^{k_1+k_2}$$

$$H^*(M) \cong H^*(N)$$ as vector spaces

but $\not\cong$ as algebras

$(\Rightarrow M \underset{\text{diff.}}{\not\cong} N)$

$M = S^2 \times S^4$ (Künneth) verify!

$N = \mathbb{CP}^3$ (via Morse theory, cell homology, Hodge theory)

$H^* \to$	0	1	2	3	4	5	6
M	$\mathbb{R}$	0	$\mathbb{R}$ ω_{S^2}	0	$\mathbb{R}$ ω_{S^4}	0	$\mathbb{R}$ $\omega_{S^2} \wedge \omega_{S^4}$
N	$\mathbb{R}$	0	$\mathbb{R}$ ω_{FS}	0	$\mathbb{R}$ ω_{FS}^2	0	$\mathbb{R}$ (ω_{FS}^3)

taken for granted

Fubini-Study

$\omega_{FS}^4 = 0$

[$\{\omega^k\}$ integral invariants of Poincaré - Cartan]

$[\omega_{FS}] = a$ $\qquad$ $[\omega_{S^2}] = a'$

$[\omega_{FS}]^2 = [\omega_{FS}][\omega_{FS}] = a^2$ $\qquad$ $[\omega_{S^4}] = b'$

every algebra isomorphisms yields without loss of generality ("wlog")

$a \mapsto a'$ but this implies

$a^2 \mapsto a'^2 = 0$

$\neq$

0

but this is impossible for an isomorphism