Intersection theory

INTERSECTION THEORY

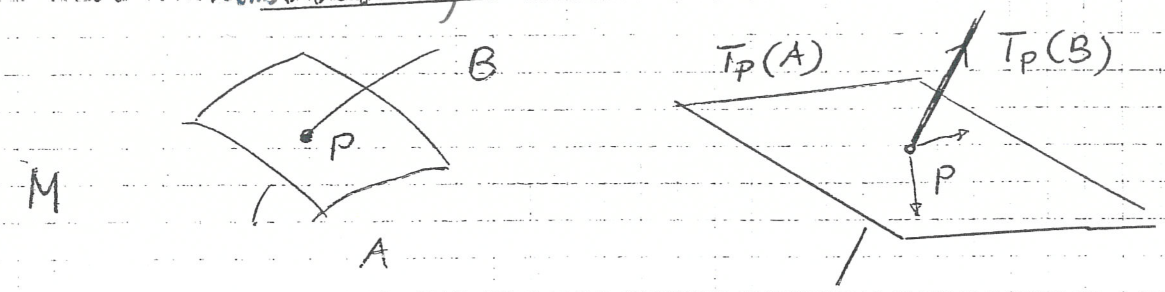
POINCARÉ DUALITY REVISITED

M smooth oriented compact manifold, of dimension n

A piecewise smooth cycle $\dim A = k$

B " $\dim B = n - k$

Let $p \in A \cap B$ such that A e B intersect transversally



Let $(v_1, \dots v_k) \in T_p(A) \subseteq T_p(M)$

oriented basis

$(w_1, w_2 \dots w_{n-k}) \in T_p(B)$

$$i_p(A \cdot B) = \begin{cases} +1 & \diamond \\ -1 & \text{otherwise} \end{cases}$$

intersection index of A and B in p

$\diamond$ if $(v_1, \dots, v_k, w_1, \dots, w_{n-k})$

is an oriented basis for $T_p(M) = T_p(A) \oplus T_p(B)$

If A and B intersect transversally everywhere one sets

$$\boxed{\#(A\cdot B) = \sum_{p\in A\cap B} i_p(A\cdot B)}$$

intersection number

☆ The above sum is finite since $A\cap B$ is discrete and A, B are compact.

Let us show that $\#(A\cdot B)$ depends only on $[A]$ e $[B]$ in $H_k(M)$ e $H_{n-k}(M)$

It is then enough to show that

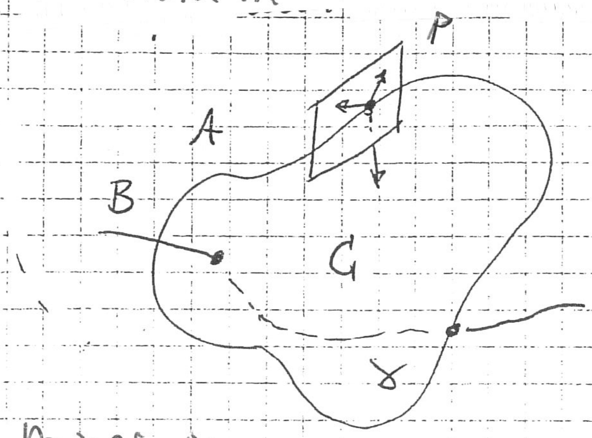
$$A \sim 0 \Rightarrow \#(A\cdot B) = 0$$

$$A = \partial G$$

Let $A = \partial C_i$ = sum of boundaries of piecewise smooth submanif. of dim $k+1$, C_i, such that $\forall$ p in A, one has an oriented basis $(v_1, \dots v_k)$ of $T_p(A)$ together with a normal vector pointing in the interior of C_i, in order to get an orientation on C_i

. one can use a Riemannian metric in M

in M)



One can arrange things in such a way that C meets B in a set $\{\gamma_\alpha\}$ of piecewise smooth arcs, whose endpoints are the intersection points of A with B

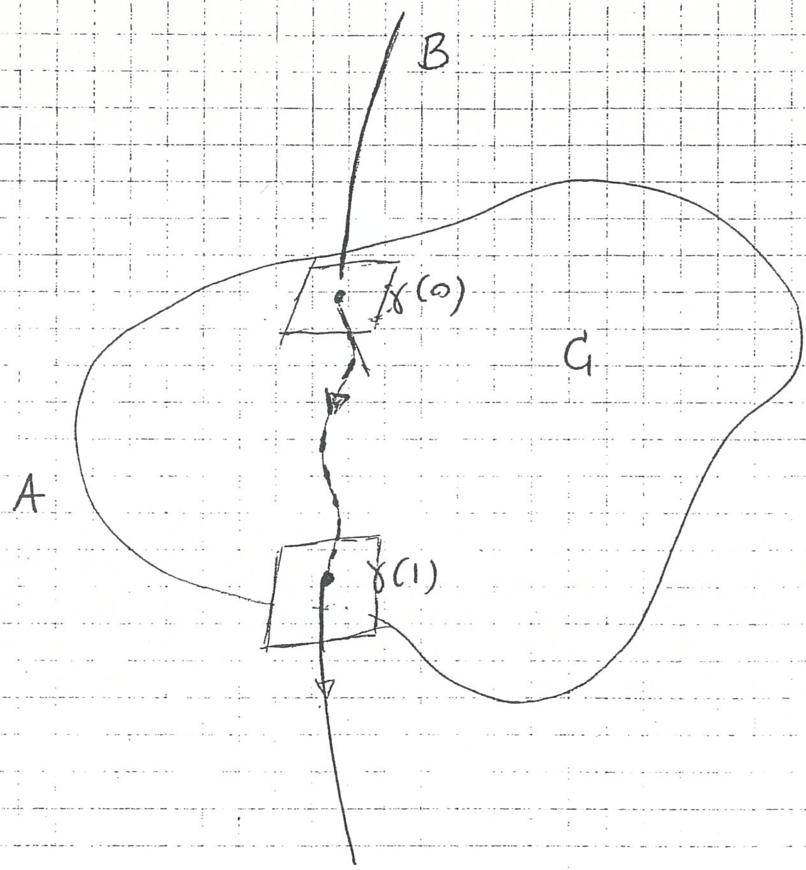
Let us show that, $\forall \gamma$, $\gamma(0)$ and $\gamma(1)$ yield opposite intersection indices

Let us find vector fields $\{v_i(t) \in T_{\gamma(t)} C\}$ $i = 1, 2, \dots, k$ and $\{v_j(t) \in T_{\gamma(t)}(B)\}_{j=k+2,\dots,n}$

such that $\forall t$

1. $v_1(t), v_2(t), \dots v_k(t), \gamma'(t)$ is an oriented basis for $T_{\gamma(t)}(C)$
2. $\gamma'(t), v_{k+2}(t), \dots v_n(t)$ is an oriented basis for $T_{\gamma(t)}(B)$
3. $v_1(t), \dots, v_k(t), \gamma'(t), v_{k+2}(t), \dots v_n(t)$ is an oriented basis for $T_{\gamma(t)}(M)$

with $v_1(0), \dots v_k(0)$ or. basis for $T_{\gamma(0)}(A)$, $v_1(1), \dots v_k(1)$ or. basis for $T_{\gamma(1)}(A)$



$\dim A = k$

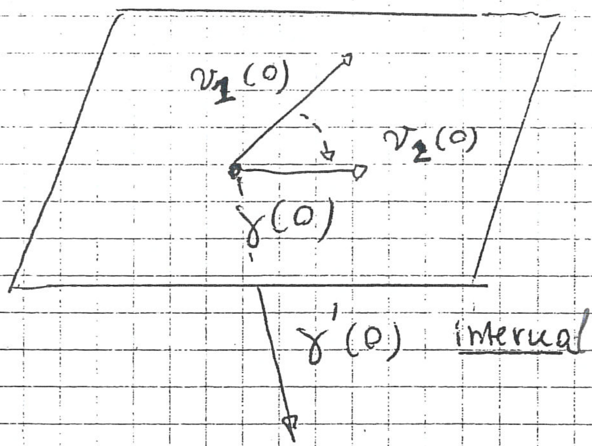
$\dim B = n - k$

$n = 3$

qui $k = 2$

$n - k = 1$

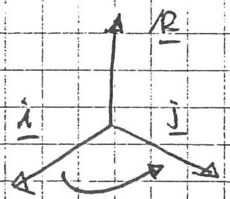
[no vectors $v_{k+1} \ldots$ present]

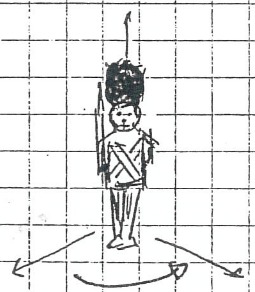


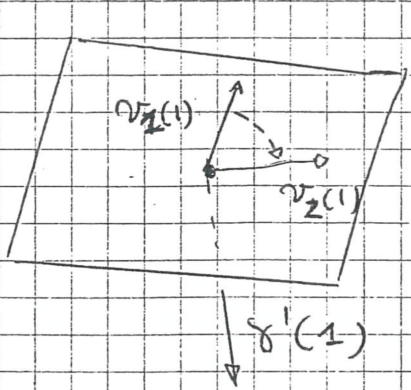
if

$(v_1(0), v_2(0))$ è positively oriented

internal







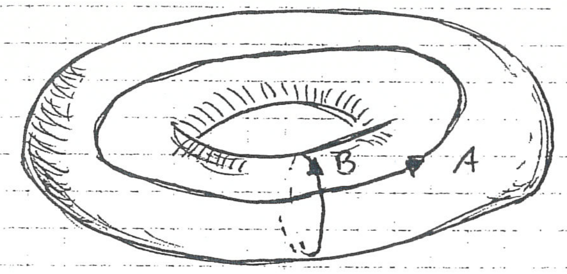
$(v_1(1), v_2(1))$ è negativ. oriented

external

This yields the conclusion

Now, given $\alpha \in H_k(M,\mathbb{Z})$, $\beta \in H_{n-k}(M,\mathbb{Z})$,
one can find A and B precise smooth cycles such that $[A] = \alpha$, $[B] = \beta$ and intersecting transversally.

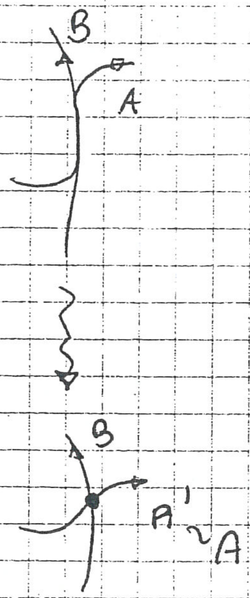
The following bilinear map (intersection pairing) is then determined:



$$H_k(M,\mathbb{Z}) \times H_{n-k}(M,\mathbb{Z}) \to \mathbb{Z}$$

$$(\alpha, \beta) = ([A],[B]) \longmapsto \#(A \cdot B)$$

transversal intersection



topological intersection of A e B, in terms of their classes

ovvero

$$\#(A \cdot B) := \#(A' \cdot B')$$

in general

$A' \sim A$
$B' \sim B$
int. trasversalmente

Theorem — <u>Poincaré duality</u>

M smooth, compact, oriented $\dim M = n$

The map

$$H_k(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$$

(intersection pairing)

is unimodular, namely, every

linear functional (:: Hom) $H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$

comes from intersection with a certain $\alpha \in H_k(M, \mathbb{Z})$, and every α having intersection number 0 with all classes is torsion