VECTOR BUNDLES V2

DIFFERENTIAL GEOMETRY & TOPOLOGY

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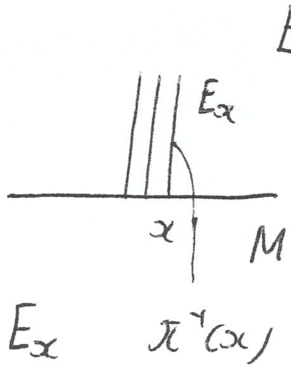
Lecture LX

☆ Vector bundles (E, M, π)

(also ...: E)

$\pi : E \to M$

canonical projection — total space — base



$\boxed{\pi^{-1}(x)}$ vector space. $\equiv E_x$
$\cap$
M

(fibre at x)

$\cong V$ "typical fibre" vector space

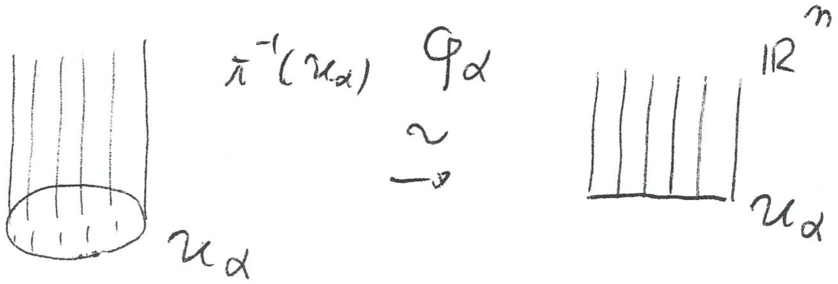
$\cong \mathbb{R}^n$

(E, M, π) : vector bundle of rank n — rank n vector bundle

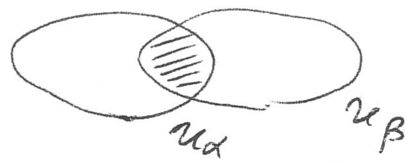
if $\exists\ \mathcal{U} = \{\mathcal{U}_\alpha\}$ of M & fibre preserving diffeomorphisms

$$\varphi_\alpha : E\Big|_{\mathcal{U}_\alpha} \equiv \pi^{-1}(\mathcal{U}_\alpha) \xrightarrow{\sim} \mathcal{U}_\alpha \times \mathbb{R}^n$$

& linear on each fibre ("local trivialization")



Consequently, the maps:



$$\varphi_\alpha \circ \varphi_\beta^{-1} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \bar{V} \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \bar{V}$$

are vector space automorphisms on each fibre

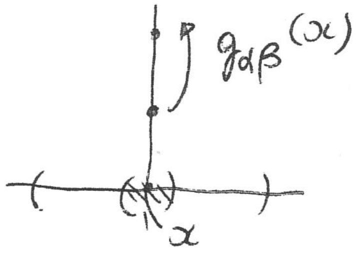
$\Rightarrow$

$$\exists \quad g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to Gl(\bar{V})$$

<u>transition</u> <u>functions</u>

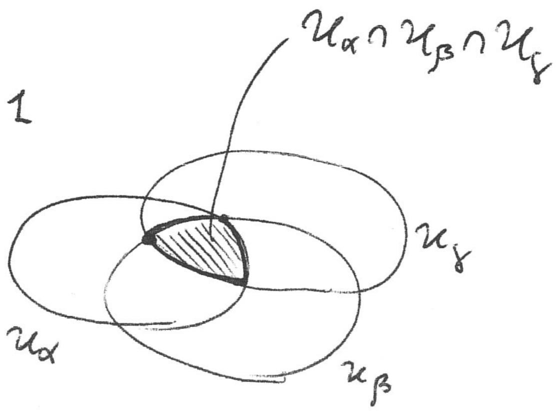
<u>invertible</u> <u>linear transformations</u>

$$\boxed{g_{\alpha\beta}(x) = \varphi_\alpha \circ \varphi_\beta^{-1} \Big|_{\{x\} \times \bar{V}}}$$



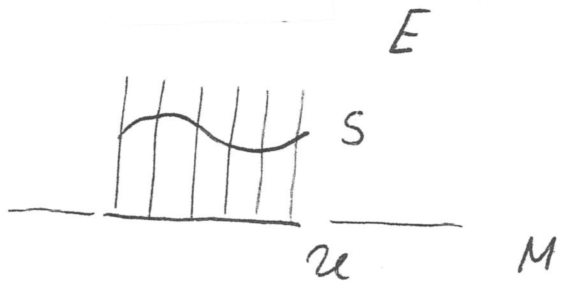
The following <u>cocycle</u> <u>conditions</u> hold

$$\boxed{\begin{cases} g_{\alpha\alpha} = 1 \\ g_{\beta\alpha} = g_{\alpha\beta}^{-1} \\ g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1 \end{cases}} \qquad \forall x \in M$$



$$\left[\, S : \mathcal{U} \overset{\subset M}{} \to E \right.$$ <u>local section</u>

$$\pi \circ S = i$$



☆ $\Gamma(\mathcal{U}, E)$ = <u>vector space of local sections</u>
(actually: it is a $\mathcal{C}^\infty(\mathcal{U})$ - <u>module</u> ..
see below)

$\left[\, (S_1 \dots S_n) \right.$ <u>local</u> <u>frame</u> : $(S_1(x) \dots S_n(x))$ <u>basis</u> of E_x

$\Gamma(E)$ = <u>global</u> <u>sections</u> of E
(it is a $\mathcal{C}^\infty(M)$-module)

$\Downarrow$

s

collection of local sections

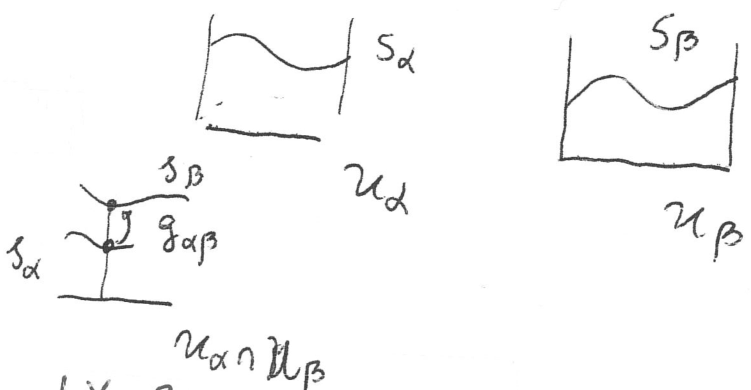
$$s = \{ s_\alpha \}$$

gluing conditions

$$\boxed{s_\beta(x) = g_{\beta\alpha}(x)\, s_\alpha(x)}$$

$s_\alpha \in \Gamma(\mathcal{U}_\alpha, E)$ $\qquad x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$

glue:



[Aside: Swan's theorem ...]

$\Gamma(E)$ $\equiv$ $\mathcal{C}^\infty(M)$ - finitely generated

smooth sections of vector bundles — projective modules

(categorial equivalence)

Γ $M = \sum_i A x_i$

A-module (left) — generators

$i \in I$ finite index set

M free: $M \cong A^n$

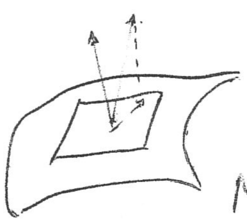
* M finit. generated $\Longleftrightarrow$ $M \cong A^n/\sim$

(quotient of a free module)

* projective: $\Gamma(E) = P\, \mathcal{C}^\infty(M)^n$

$P \in \mathrm{End}\, \mathcal{C}^\infty(M)^n$ $\quad (\equiv \{P_x\}_{x \in M})$

$P = P^* = P^2$ projection



M

[## Upshot

One can define the concept of (smooth) vector bundle on a C^* algebra ...

$\rightsquigarrow$ noncommutative (differential) geometry]

Operations on vector bundles $\pi: E \to M$

We describe them directly in terms of transition functions (for a more intrinsic approach see Atiyah: "K-Theory")

① $E \oplus F$ — direct sum

fibre $(E \oplus F)_x = E_x \oplus F_x$

if $\{\mathcal{U}_\alpha\}$ trivialises both bundles, we get

$$\underbrace{t_{\alpha\beta}(x)}_{E \oplus F} = \begin{pmatrix} \overbrace{g_{\alpha\beta}(x)}^{E} & 0 \\ 0 & \underbrace{h_{\alpha\beta}(x)}_{F} \end{pmatrix}$$

② $E \otimes F$ — tensor product

$(E \otimes F)_x = E_x \otimes F_x$

$$\underbrace{t_{\alpha\beta}(x)}_{E \otimes F} = g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x)$$

③ E^* — dual bundle

fibre: E_x^*

$$g^*_{\alpha\beta}(x) = \left[g_{\alpha\beta}(x)^t\right]^{-1}$$

ex: $E = TM$ tangent bundle
$E^* = T^*M$ cotangent bundle (verify!)

$$\mathrm{Hom}(E, E') \cong E^* \otimes E'$$

$\Lambda^k(E)$ k-th exterior power

ex: $\boxed{\Omega^k(M) = \Gamma(\Lambda^k T^* M)}$

k-forms sections of

equivalent cocycles

$g_{\alpha\beta} \longleftrightarrow \varphi_\alpha$

$g'_{\alpha\beta} \longleftrightarrow \varphi'_\alpha$

$$\varphi_\alpha = \underset{\substack{\uparrow \\ GL(n,\mathbb{R})}}{\lambda_\alpha^{(x)}} \varphi'_\alpha$$

$$\boxed{\begin{aligned} g_{\alpha\beta} = \varphi_\alpha \cdot \varphi_\beta^{-1} &= \lambda_\alpha \varphi'_\alpha \varphi'^{-1}_\beta \lambda_\beta^{-1} \\ &= \lambda_\alpha g'_{\alpha\beta} \lambda_\beta^{-1} \end{aligned}}$$

If E is trivial one may set $g'_{\alpha\beta} \equiv 1$

$\Rightarrow$ in general $g_{\alpha\beta} = \lambda_\alpha \lambda_\beta^{-1}$

("cocycle") ("coboundary")

$\Rightarrow$ E is trivial

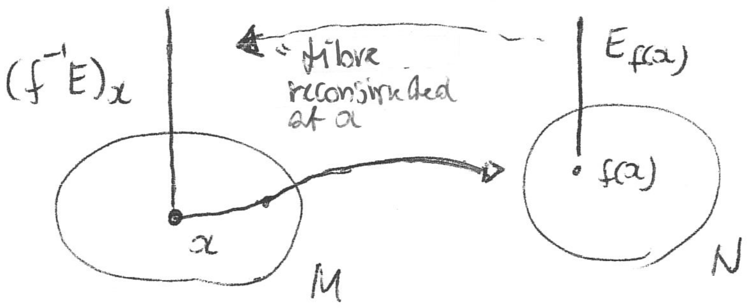
$f : E \to F$ <u>bundle</u> <u>map</u> (over the same base manifold M)

f sends <u>fibres into fibres</u> and it is <u>linear</u> thereon (that is, on corresponding fibres)

Let $f : M \to N$

given $\pi : E \to N$, one defines the <u>pulled</u>-<u>back</u> <u>bundle</u> (<u>pull-back of E</u>) $f^{-1}E$ over M as follows (or f^*E)

$$(f^{-1}E)_x := E_{f(x)}$$



(it is indeed <u>locally</u> <u>trivial</u>)

E, vector bundle, is termed <u>orientable</u>

if its <u>structure group</u> $(GL(n, \mathbb{R}))$

can be <u>reduced</u> to $GL^+(n, \mathbb{R})$ that is
(positive determinant)

the $\{g_{\alpha\beta}(x)\}$ <u>can be chosen</u> in

$GL^+(n, \mathbb{R})$ $(\det > 0)$

<u>Prop.</u> An <u>orientable</u> vector bundle E over an orientable manifold is orientable (as a manifold)

<u>Proof</u>

M $\{\mathcal{U}_\alpha, \Psi_\alpha\}$ <u>oriented</u> <u>atlas</u>

$\dim M = m$ $h_{\alpha\beta} := \Psi_\alpha \circ \Psi_\beta^{-1}$ transition functions

$$\phi_\alpha : E|_{\mathcal{U}_\alpha} \xrightarrow{\sim} \mathcal{U}_\alpha \times \mathbb{R}^n$$

transition functions $g_{\alpha\beta}(x)$

$$E|_{\mathcal{U}_\alpha} \overset{\phi_\alpha}{\xrightarrow{\sim}} \mathcal{U}_\alpha \times \mathbb{R}^n \overset{\Psi_\alpha}{\xrightarrow{\sim}} \mathbb{R}^m \times \mathbb{R}^n$$

$(\mathcal{U}_\alpha, \Psi_\alpha \circ \phi_\alpha)$ <u>atlas for</u> E

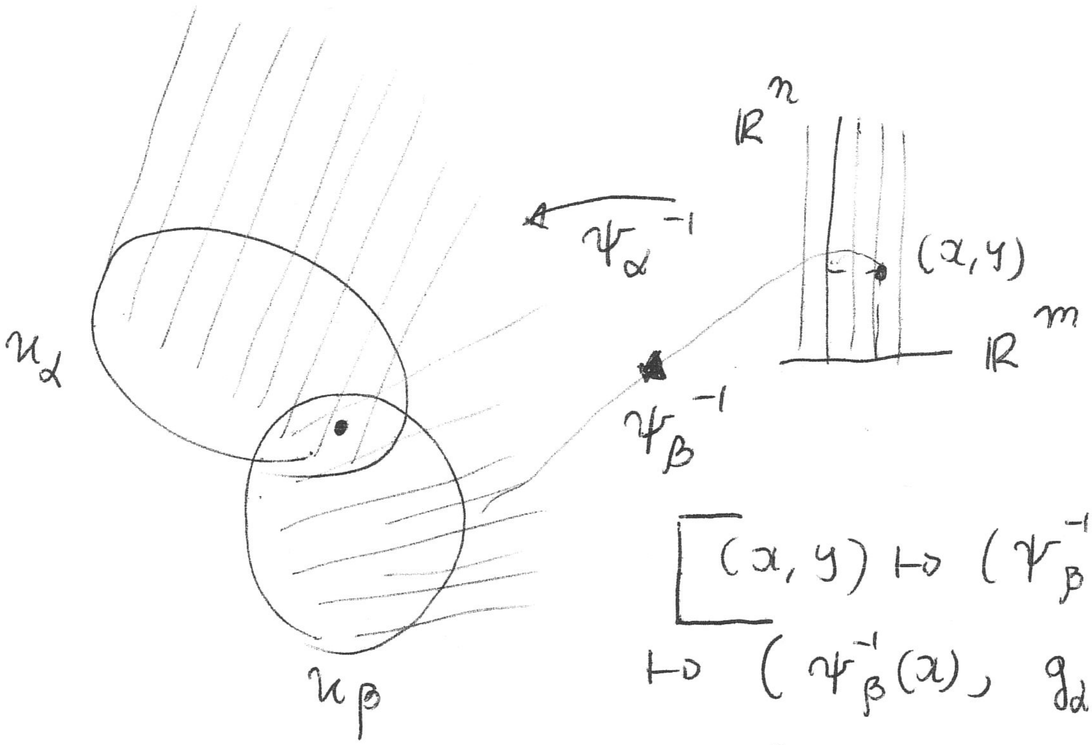
Transition functions :

$$(\psi_\alpha \times 1) \circ \underbrace{\phi_\alpha \phi_\beta^{-1}}_{g_{\alpha\beta}} \circ (\psi_\beta^{-1} \times 1) : \mathbb{R}^m \times \mathbb{R}^n \circlearrowleft$$

$$(x, y) \longmapsto \left(h_{\alpha\beta}(x) , g_{\alpha\beta}(\psi_\beta^{-1}(x)) y \right)$$

Jacobian matrix
$$\begin{pmatrix} J(h_{\alpha\beta}) & * \\ 0 & g_{\alpha\beta}(\psi_\beta^{-1}(x)) \end{pmatrix}$$

having $\det > 0$ $\square$



$$(x, y) \mapsto (\psi_\beta^{-1}(x), y) \mapsto$$
$$\mapsto (\psi_\beta^{-1}(x), g_{\alpha\beta}(\psi_\beta^{-1}(x)) y)$$
$$\mapsto (h_{\alpha\beta}(x), g_{\alpha\beta}(\psi_\beta^{-1}(x)) y$$