LV2 DIFFERENTIAL GEOMETRY & TOPOLOGY

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Lecture LXI

The Thom isomorphism theorem

$$\left[H^*(M) \overset{\tau}{\xrightarrow{\sim}} H_{cv}^{*+n}(E) \right]$$

(of finite type) orientable

compact vertical cohomology

$\pi : E \to M$

(compact vertical)

$$\left[\pi_* : \Omega_{cv}^*(E) \to \Omega^{*-n}(M) \right]$$ fibre integration

[compactness intervenes only in vertical directions]

- THOM ISOMORPHISM
- POINCARÉ DUAL & THOM CLASS
- EULER CLASS

Explicitly

$$\left[\tau(\bullet) = \pi^*(\bullet) \wedge \Phi \right]$$

Thom isomorphism

$\Phi = \tau(1)$

$\Phi \equiv \Phi(E)$: Thom class of E $\in H_c^n(E)$, $n =$ rank of E

$$\left[\tau = \pi_*^{-1} \right]$$

(in cohomology)

(cf. the special case already discussed !)

$$\left[\Phi \Big|_{\text{fibre}} = \text{generator of } H_c^n(F) \right]$$

Fibre

$n =$ rank of E

Special case

$$\left[H_{cv}^*(M \times \mathbb{R}^n) \simeq H^{*-n}(M) \right]$$

" Poincaré lemma "

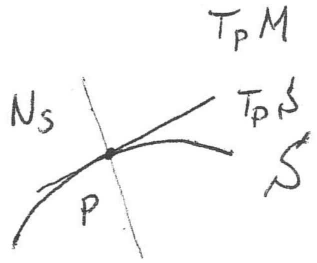
Thom = $\dfrac{\text{Poincaré lemma} + \text{Mayer-Vietoris}}{(+\ M \text{ of finite type})}$ $\square$

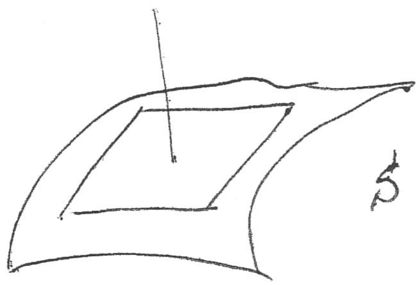
Poincaré duality and Thom class.

S closed oriented submanifold, $\dim S = k$

η_S Poincaré dual ($n-k$-form ..)

we shall interpret η_S as the Thom class of the normal bundle N of S in M.





(a metric may be introduced in order to carry out explicit operation, but the net result is metric independent)

N is determined by

$$0 \to T_S \to T_M\big|_S \to N \to 0$$

T_S: tangent bundle of S

$T_M|_S$: restriction of T_M to S

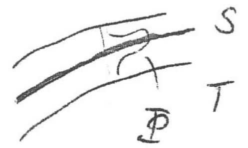
$$\underset{\substack{\cap \\ S}}{N_p} = T_pM / T_pS$$

The tubular neighbourhood theorem (proof omitted)

Every S possesses a tubular neighbourhood diffeomorphic to N_S $\square$ ($\equiv$ open neighbourhood $\supset S$ diffeomorphic to a rank $n-k$ vector bundle)

M and S orientable $\Rightarrow$ N orientable : it is then meaningful to consider the Thom class $\Phi(N)$

$$\Phi(N) \;\; \| \;\; T : \text{tube}$$



Now

$$\boxed{\int_M w \wedge j_*\Phi = \int_T w \wedge \Phi} \qquad w \in \Omega_c^k(M), \quad dw = 0$$

j_*: ext. via 0..

$i : S \to T$ inclusion

$\pi : T \to S$ retraction (by definition)

$\Rightarrow \quad i^* = \pi^{*-1}$ in cohomology.

$\Rightarrow \quad w = \pi^* i^* w + d\tau$. Hence

$$\int_T w \wedge \Phi = \int_T \pi^* i^* w \wedge \Phi + ..0 \quad \text{(Stokes)}$$

$$= \int_S i^* w$$

(via fibre integration and the very definition of Thom class)

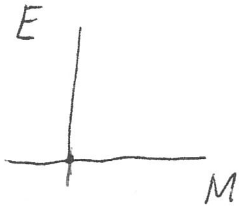
$$\Rightarrow \boxed{j_*\Phi = \eta_S}$$

The properties of η follow from the general properties satisfied by Thom

In general: $M \hookrightarrow$ 0-section in E (oriented)

$$0 \to T_M \to T_E\big|_M \to E \to 0$$

E $\|$ normal bundle to M in E ⚠

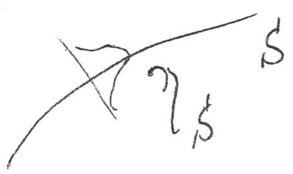


$$\Rightarrow \boxed{\eta_{S_0(M)} = \Phi(E)}$$

Finally, one has the following

Localization principle

The support of η_S can be taken in an arbitrary tubular neighborhood T of S ($N \simeq T$ arbitrary)



The Euler class (for a vector bundle)

Let $\pi : E \to M$ be an oriented vector bundle of rank k

$\left[\text{and } \Phi(E) \text{ its Thom class} \right]$

$\Phi(E) \in H^k_{cv}(E)$

Let $M \xrightarrow{s_0} E$ viewed as the zero section. Define

$$\boxed{\Phi(E)\big|_{s_0(M)} = s_0^* \Phi(E) \in H^k(M)}$$

$$s_0^*\Phi(E) \equiv e(E) \text{ of } E$$

Euler class

important special case $E = TM$

$$\boxed{e(TM) \in H^n(M)}$$

Euler number: $\boxed{\int_M e(TM)}$ M compact — well-defined

Fact (v. Bott-Tu)

Let M oriented & compact, $\dim M = k$,

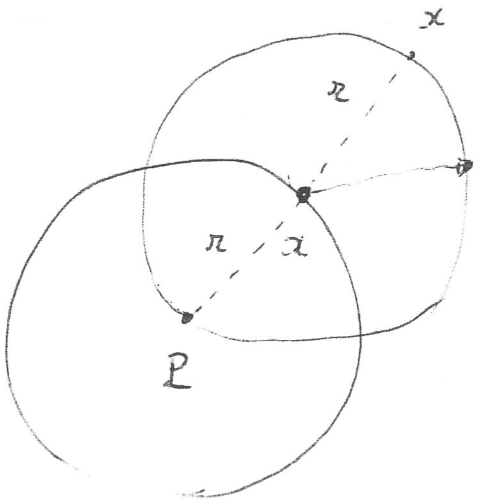
$\pi : E \to M$ di rango k.

Let s be a section with a finite number of zeros. Then:

$$\boxed{e(E) = \sum_{\text{zeros of } s}}$$

counted with their multiplicity ~ local degree

Let us examine the case $E = TM$



$f(x) = \frac{X}{\|X\|}$ vector field

(a Riemannian metric is introduced)

$X(P) = 0$ isolated zero

$r > 0$ sufficiently small

[index of X in P : mult. of P (zero of X) $=$ $\deg f$]

[$f: S^{n-1} \to S^{n-1}$]

$$\deg f = \int_{S^{n-1}} f^* \alpha \quad \in \mathbb{Z}$$

$[\alpha]$ generates $H^{n-1}(S^{n-1}) \cong \mathbb{R}$

$$\int_{S^{n-1}} \alpha = 1$$

(α : volume form)

[one works in general with proper maps $f: \mathbb{R}^n \to \mathbb{R}^n$ and with $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$ ...]

[Aside : If E is trivial, $e(E) = 0$: indeed there exists a never vanishing section, and one applies the preceding result. (indeed, k sections)

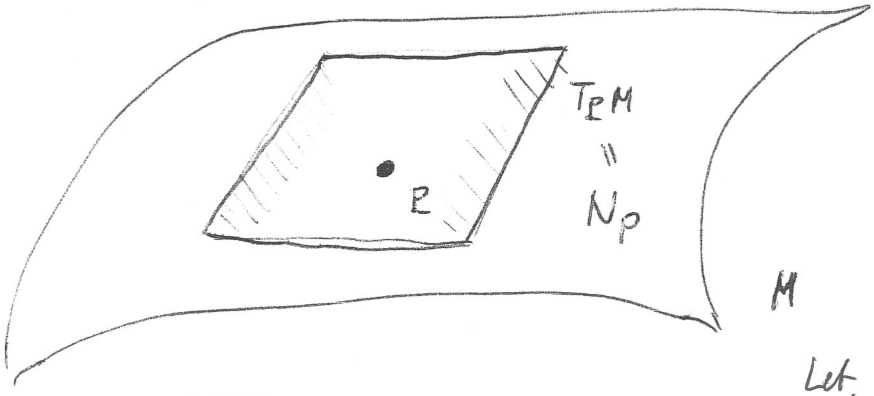
The converse is false : ex : S^5 : $\chi(S^5) = 0$ but S^5 is not parallelizable

Apply Euler-Poincaré ($\dim M = 2k+1 \Rightarrow \chi(M) = 0$)

(only S^1, S^3, S^7 are such)

* sketch of proof of ♣

for $E = TM$



p : isolated zero of a vector field (section of TM)

Let * mult. $p = 1$

☆ Normal bundle of p

$$N_p = T_pM \qquad \text{(a unique fibre !)}$$

$$\eta_p = \underset{\text{Thom}}{\Phi}(N_p) = \Phi(T_pM)$$

$\cap$

$H^n(M)$

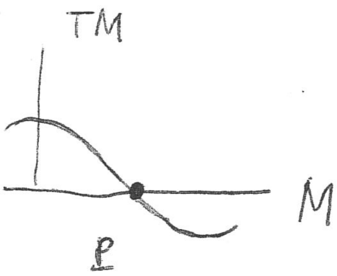
$\parallel$

generator of $H^n_c(T_pM)$

$[\alpha]$

$\parallel$

$H^n_c(\mathbb{R}^n)$



$$\eta_{k\,p} = \underset{\text{degree}}{k}[\alpha]$$

$$\Rightarrow \boxed{\eta_{\sum k_i p_i} = e(TM)}$$

M compact

$$\int_M e(TM) = \int_M \eta_{\sum k_i p_i} = \sum_i k_i \underbrace{\int \eta_{p_i}}_{=1} = \sum_i k_i$$