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Lecture LXII

- EULER CHARACTERISTIC = EULER NUMBER
- THE POINCARÉ - HOPF THEOREM

☆ **Theorem** Let M be a compact oriented manifold without boundary. Then one has

Euler characteristic = Euler number

$$\boxed{\chi(M) = \int_M e(TM)}$$

$$\chi(M) = \sum (-1)^q h^q(M)$$

Proof Let $\{w_i\}$ be a basis of $H^*(M)$ (finite dimension). Let $\{\tau_i\}$ be the dual basis (with respect to Poincaré duality):

abuse of notation...

$$☆ \quad \int_M w_i \wedge \tau_j = \delta_{ij}$$

Let

$$\begin{tikzcd} & M \times M \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ M & & M \end{tikzcd}$$

be the two projections

By virtue of the ☆ Künneth formula

$$H^*(M \times M) = H^*(M) \otimes H^*(M)$$

$\{\pi^* w_i \wedge p^* \tau_j\}$ is a basis of $H^*(M \times M)$

Then

$$\eta_\Delta = \sum c_{ij}\, \pi^* w_i \wedge \rho^* \tau_j$$

Poincaré dual
of the diagonal Δ

$\Delta \hookrightarrow M \times M$

Actually:

$$\boxed{\eta_\Delta = \sum (-1)^{\deg w_i}\, \pi^* w_i \wedge \rho^* \tau_i}$$

which is obtained as follows. Let us compute

$$(*) \quad \boxed{\int_\Delta \pi^* \tau_k \wedge \rho^* w_\ell}$$

in two different ways

Let $\quad i : M \to \Delta \subset M \times M$

$$(*) = \int_M i^* \pi^* \tau_k \wedge i^* \rho^* w_\ell =$$

$$= \int_M \tau_k \wedge w_\ell = (-1)^{\deg \tau_k \cdot \deg w_\ell}\, \delta_{k\ell}$$

But one also has, by the very definition of Poincaré dual

$$(*) = \int_{M\times M} \pi^*\tau_k \wedge \rho^* w_\ell \wedge \eta_\Delta$$

$$= \sum_{i,j} c_{ij} \int_{M\times M} \pi^*\tau_k \wedge \rho^* w_\ell \wedge \pi^* w_i \wedge \rho^* \tau_j$$

$$= \sum c_{ij} (-1)^{(\deg \tau_k + \deg w_\ell)\deg w_i} \cdot$$

$$\cdot \int_{M\times M} \pi^*(w_i \wedge \tau_k)\, \rho^*(w_\ell \wedge \tau_j)$$

$$= (-1)^{(\deg \tau_k + \deg w_\ell)\deg w_k} \quad c_{k\ell}$$

$$\Rightarrow \quad c_{k\ell} = \begin{cases} 0 & k \neq \ell \\ (-1)^{\deg w_k} & k = \ell \end{cases}$$

$$\left(c_{k\ell} = (-1)^{\deg w_k} \delta_{k\ell} \right)$$

Let us now show that

$$\boxed{N_\Delta \cong T_M}$$

One has, since $i : M \to M \times M$ is a diffeomorphism on Δ, $i^* T_\Delta = T_M$

Let us consider the commutative diagram

$(v,v) \mapsto (v,v)$ $\quad\quad$ ✱ normal bundle

$$\begin{array}{ccccccccc}
 & 0 & \to & T_\Delta & \to & T_{M\times M}|_\Delta & \to & N_\Delta & \to & 0 \\
\Delta \cong M \implies & & & \wr\wr & & \wr\wr & & & & \\
 & 0 & \to & T_M & \to & T_M \oplus T_M & \to & T_M & \to & 0
\end{array}$$

$v \longmapsto (v,v)$

$\Rightarrow$ the conclusion $\boxed{N_\Delta \cong T_M}$

Now

$$\int_\Delta \eta_\Delta = \int_\Delta \Phi(N_\Delta)$$

Thom class of the normal bundle N_Δ viewed as a tubular neighbourhood of Δ in $M \times M$

$$= \int_\Delta e(N_\Delta) \quad ✱ \quad \left(\Phi|_{0\text{-section}} = e \right)$$

Euler class

$$= \int_M e(T_M)$$

Eventually

$$\int_\Delta \eta_\Delta = \underbrace{\int_M e(T_M)}_{\text{Euler number}}$$

☆ Poincaré duality $\quad \|$

$$\int_{M\times M} \eta_\Delta \wedge \eta_\Delta$$

★ Self-intersection number of Δ in $M\times M$

Let us compute $\int_\Delta \eta_\Delta$ via the preceding formulae:

$$\int_\Delta \eta_\Delta = \sum_i (-1)^{\deg w_i} \int_\Delta \pi^* w_i \wedge p^* \tau_i =$$

$$= \sum_i (-1)^{\deg w_i} \int_M i^*\pi^* w_i \wedge i^* p^* \tau_i =$$

$$= \sum_i (-1)^{\deg w_i} \underbrace{\int_M w_i \wedge \tau_i}_{=1}$$

$$☆ = \sum_q (-1)^q \underset{\underset{b_q(M)}{\|}}{h^q(M)} = \boxed{\chi(M)}$$

□

An immediate corollary of the previous calculation is the

~~the~~ Poincaré – Hopf theorem

Let M be a compact oriented manifold.

Let X be any vector field with isolated zeros

(a finite number...)

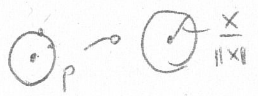
Then

$$\boxed{\sum_i i_{p_i}(X) = \chi(M)}$$

|| def. via a Riemannian metric

index of X in p_i isolated zero

i_p: local degree of $\frac{X}{\|X\|}$



Remember
$\sum i_{p_i}(X) =$
$\int_M \sum k_i \eta_{p_i}$
$\sum k_i \int_M \eta_{p_i} =$
$\sum k_i$
"
$\int_M e(TM)$

Indeed $\sum_i i_{p_i}(X) = \int_M e(TM) = \chi(M)$ by the preceding result

Corollary: There does not exists a never vanishing vector field on S^2 : $\chi(S^2) = 2 \neq 0$!

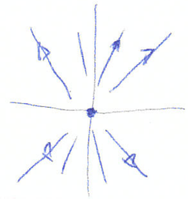
" one cannot comb a tennis ball "

Corollary: TS^2 is a non trivial bundle

(but N_{S^2} ($S^2 \hookrightarrow \mathbb{R}^3$) is trivial..)

Examples

$$\begin{cases} \dot{x} = x & = X \\ \dot{y} = y & = Y \end{cases} \qquad \begin{cases} x = x_0 e^t \\ y = y_0 e^t \end{cases}$$

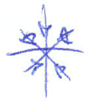


unstable node

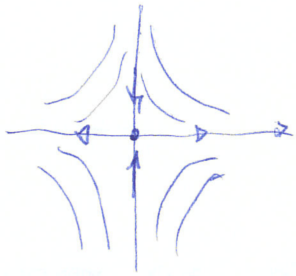
$$i_0 = \frac{1}{2\pi} \oint \frac{X\,dY - Y\,dX}{X^2+Y^2} = \frac{1}{2\pi} \oint \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= \dots +1$$

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$$ stable node



$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}$$



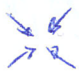
saddle point, mountain pass

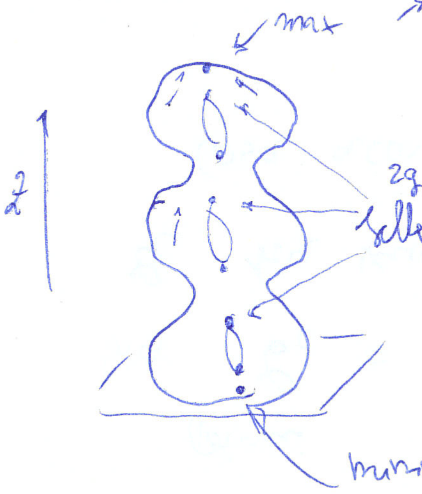
$$x = x_0 e^t \qquad y = y_0 e^{-t}$$

$$i_0 = \frac{1}{2\pi} \oint \frac{-x\,dy + y\,dx}{x^2+y^2} = -1$$

$\boxed{\text{Application}}$

Σ_g

 $+1$



f: height function

$X = \nabla f$

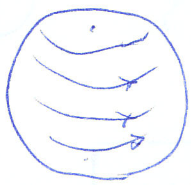
[Poincaré - Hopf]

$$\chi(\Sigma_g) = \sum i_{p_i}(\nabla f) = 2 - 2g$$

 -1

 $+1$

$i(X) = 2$



$i(X) = 0$ (no critical pts)

idem

The Euler - Poincaré Theorem

$$\boxed{V - E + F = 2}$$

vertici facce

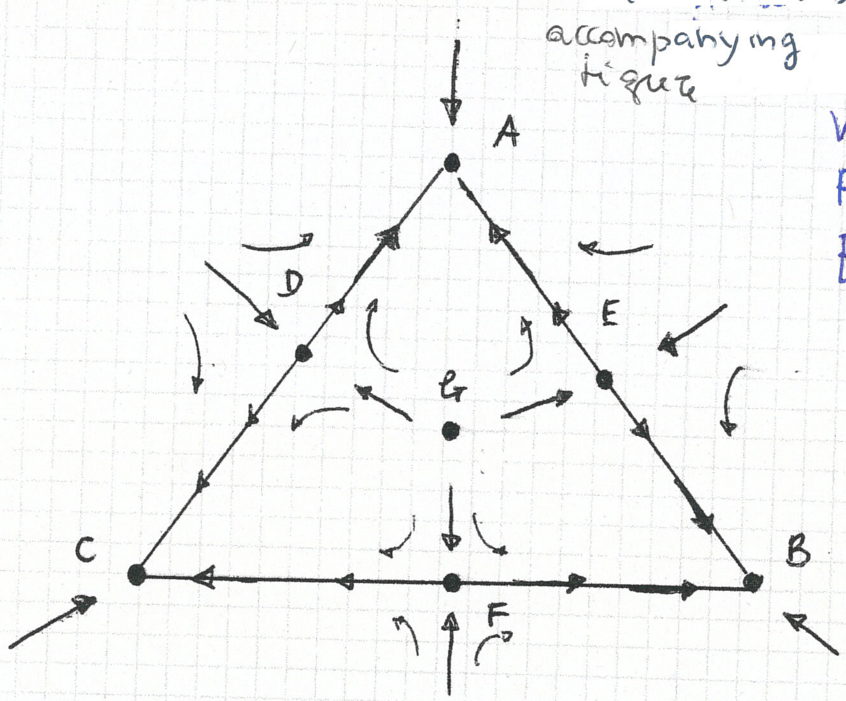
spigoli

$(M \simeq S^2)$

Proof via

Poincaré - Hopf

Key idea   One starts from $\chi(S^2) = 2$ and constructs X as in the accompanying figure



One finds:

V stable nodes $(+1)$

F unstable nodes $(+1)$

E saddle pts (-1)

$$i(X) = V - E + F$$
$$= \chi(S^2)$$
$$= 2$$

consider the vector field in figure,

con singolarities $A, B \dots F, G$

$i_G = i_A = i_B = i_C = +1 \qquad i_D = i_E = i_F = -1$