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Lecture LXIII

★★★ The Maurer-Cartan form

Let G be a Lie group (fin. dim.) and $\mathfrak{g}$ its Lie algebra

Let (Λ_a) be a basis of $\mathfrak{g}$ $\quad a = 1, 2, \dots \dim \mathfrak{g} = \dim G$

$$\underbrace{[\Lambda_a, \Lambda_b]}_{\text{bracket}} = f^{\,c}_{ab}\,\Lambda_c$$

$\{f^{\,c}_{ab}\}$ structure constants of $\mathfrak{g}$

(invariant by Lie algebra automorphisms)

Define, on G, the $\mathfrak{g}$-valued 1-form (MC-form)

$$g \longmapsto g^{-1}dg =: \Phi$$

$\boxed{\star \quad \Phi \text{ is } G\text{-invariant}}$: let $g_0 \in G$ fixed

Then: $$(g_0 g)^{-1} d(g_0 g) = g^{-1} \underbrace{g_0^{-1} g_0}_{= e} dg = g^{-1}dg$$

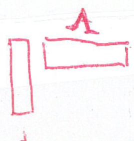
Write: $$\Phi = g^{-1}dg = \phi^a \Lambda_a$$

ϕ^a: left-invariant 1-forms providing a basis dual to (Λ_a)
$\phi^a(\Lambda_b) = \delta^a_b$

We have Cartan's structure equation

$$\boxed{d\phi^a + \frac{1}{2} f^{\,a}_{bc}\, \phi^b \wedge \phi^c = 0}$$

Let us give two proofs of this formula



$\Phi = \phi \otimes \Lambda$

$n\times 1 \quad 1 \times n$

$n \times n$

First compute $d(g^{-1}dg) = dg^{-1} \cdot dg$

From $0 = d(g^{-1}g) = dg^{-1}g + g^{-1}dg = 0$

$$dg^{-1} = -g^{-1}dg\,g^{-1}$$

we have

$$d(\underbrace{g^{-1}dg}_{\Phi}) = -(\underbrace{g^{-1}dg}_{\Phi})(\underbrace{g^{-1}dg}_{\Phi}) \quad \text{i.e.}$$

$$\boxed{d\Phi + \Phi \wedge \Phi = 0}$$

for matrix valued forms, just multiply matricially, taking wedge product for form entries. See next comment for dealing with the general case

Explicitly, if

$$\Phi = \phi^a \Lambda_a \qquad (\phi^a \text{ left invariant dual to } \Lambda_a)$$

$$0 = d\Phi + \Phi \wedge \Phi = d\phi^a \Lambda_a + \phi^b \Lambda_b \wedge \phi^c \Lambda_c =$$

$$d\phi^a \cdot \Lambda_a + \phi^b \wedge \phi^c \Lambda_b \Lambda_c \overset{!}{=} d\phi^a \cdot \Lambda_a + \frac{1}{2} \phi^b \wedge \phi^c (\Lambda_b \Lambda_c - \Lambda_c \Lambda_b)$$

$$d\phi^a \cdot \Lambda_a + \frac{1}{2} \phi^b \wedge \phi^c \underbrace{[\Lambda_b, \Lambda_c]}_{f^a_{bc}\Lambda_a} =$$

$$\left[d\phi^a + \frac{1}{2} f^a_{bc}\, \phi^b \wedge \phi^c\right] \Lambda_a$$

$\Rightarrow$ we get Cartan's structure equation.

Now, let us compute in the following manner

First:

$$\boxed{d\phi^a(\Lambda_b,\Lambda_c)} = \Lambda_b \underbrace{\phi^a(\Lambda_c)}_{\delta^a_c} - \Lambda_c \underbrace{\phi^a(\Lambda_b)}_{\delta^a_b} - \phi^a([\Lambda_b,\Lambda_c])$$

as a l. inv. v. field

$$= -\phi^a(f^d_{bc}\Lambda_d) = -f^d_{bc}\underbrace{\phi^a(\Lambda_d)}_{\delta^a_d} = \boxed{-f^a_{bc}}$$

Second:

$$\boxed{\tfrac{1}{2}f^a_{b'c'}\phi^{b'}\wedge\phi^{c'}(\Lambda_b,\Lambda_c)} = \tfrac{1}{2}f^a_{b'c'}\Big[\underbrace{\phi^{b'}(\Lambda_b)}_{\delta^{b'}_b}\underbrace{\phi^{c'}(\Lambda_c)}_{\delta^{c'}_c} - \underbrace{\phi^{b'}(\Lambda_c)\phi^{c'}(\Lambda_b)}_{\delta^{b'}_c\,\delta^{c'}_b}$$

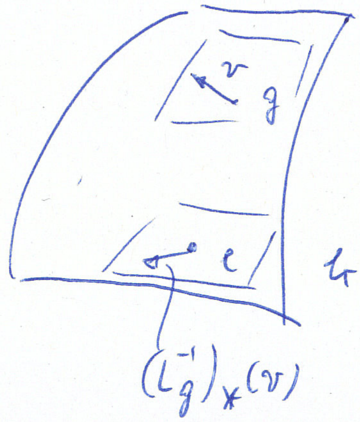
$$= \tfrac{1}{2}f^a_{bc} - \tfrac{1}{2}\underbrace{f^a_{cb}}_{= -f^a_{bc}} = \boxed{+f^a_{bc}}$$

Therefore $\boxed{d\phi^a + \frac{1}{2}f^a_{bc}\phi^b\wedge\phi^c = 0}$, again. □

☆ Intrinsic definition of Φ

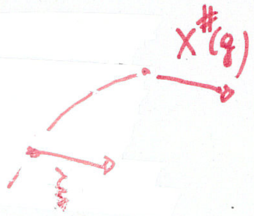
$$\Phi : \quad T_gG \longrightarrow \Phi(v) \in T_eG$$

$$\Phi(v) := (L_g^{-1})_*(v)$$



If $X^\# \in \mathfrak{g}$ and $X^\#(e) = \xi$, then

$$\boxed{\Phi(X^\#(g)) = \xi \quad \forall\, g\in G}$$



$X^\#(g) = (L_g)_*\xi \qquad \Phi(X^\#(g)) = \dots \xi$

Conundrum

We have just seen that, if $\mathfrak{g}$ is a matrix Lie algebra (actually, this is general in view of Ado's theorem), then in order to multiply two matrix valued forms, we just multiply matrices as usual, using the wedge product for their entries. But, abstractly, we have just the bracket $[\,,\,]$, and not a product wherefrom $[\,,\,]$ could come.

One defines

$$[\underset{\substack{\cap\\ \Lambda^k(M)}}{\omega} \otimes \underset{\substack{\cap\\ \mathfrak{g}}}{X}, \overset{\substack{\Lambda^h(M)\\ \cup}}{\varphi} \otimes \underset{\substack{\cap\\ \mathfrak{g}}}{Y}] := \omega \wedge \varphi \otimes [X, Y]$$

(then one extends...)

Now, let us retrieve the MC-form $\Phi = \phi^a \Lambda_a$

We claim that the Cartan equation becomes

$$d\Phi + \frac{1}{2}[\Phi, \Phi] = 0$$

This is clear since if $\Phi = \phi^a \Lambda_a$,

$$\underbrace{\frac{1}{2}[\phi^b \Lambda_b, \phi^c \Lambda_c]}_{\frac{1}{2}[\Phi, \Phi]} = \frac{1}{2}\phi^b \wedge \phi^c [\Lambda_b, \Lambda_c] = \underbrace{\frac{1}{2} f^a_{bc} \phi^b \wedge \phi^c \Lambda_a}_{\Phi \wedge \Phi}$$