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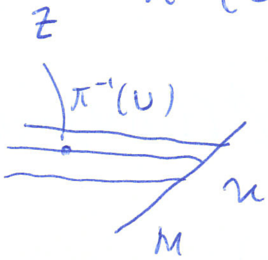
CONNECTIONS ON VECTOR BUNDLES

** Connections on vector bundles

$\pi : E \to M$ vector bundle
of rank n, fibre F (vector space)

Let $(e_1 \dots e_k)$ be a local frame
in $U \subset M$

$$\pi^{-1}(U) \ni Z = z^i e_j \quad . \quad \text{One has a trivialization}$$



$$\pi^{-1}(U) \cong U \times F \sim (x, z)$$

local coordinates

$$e_i = (0 \dots, 1, \dots 0)$$

ith place

! just a local description!

local section : $s(x) = z^i(x) e_i(x)$ $\quad s : M \to E$

$\pi \circ s = id$

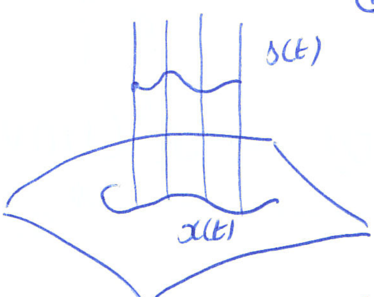
$$TE \overset{\text{locally}}{\sim} \left(\partial/\partial x^\mu , \frac{\partial}{\partial z^i} \right)$$

$$T^*E \sim \left(dx^\mu , dz^i \right)$$

* Connection : a rule ~~~~ identifying (nearby) fibres
(there is no canonical way to do it)

Define $\nabla_X s$ (and get another section)

① parallel transport: $\left\{ \nabla_{\dot{x}} s = 0 \right\}$



s : parallel transported

$$\nabla_{\partial_\mu} (e_i) = \Gamma^j_{\mu i} e_j \qquad \text{generalized Christoffel symbols}$$

from $\dfrac{df}{dt} = \dot{x}^\mu \dfrac{\partial f}{\partial x^\mu}$ we impose (Leibniz)

$$\boxed{\nabla_{\dot{x}} s} = \nabla_{\frac{d}{dt}} (e_i z^i) = \nabla_{\frac{d}{dt}} (e_i) z^i + e_j \dot{z}^j$$

$$\nabla_{\frac{d}{dt}} \equiv \frac{\nabla}{dt}$$

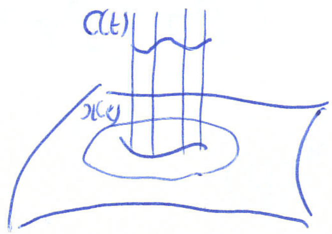
$$= \dot{x}^\mu \left[\nabla_\mu (e_i) z^i + e_j \partial_\mu z^j \right]$$

$$= \dot{x}^\mu e_j \left(\Gamma^j_{\mu i} z^i + \partial_\mu z^j \right)$$

is parallel transported along $x = x(t)$ if, by definition

$$\dot{x}^\mu \left(\Gamma^j_{\mu i} z^i + \partial_\mu z^j \right) = 0 \qquad \forall\, j = 1 \dots n$$

② A second way of looking at the same question:



Lift $x = x(t)$ in M to $c = c(t)$ in E (thus we get a section on $x(t)$) fulfilling the parallel transport equation above:

$$c(t) = \left(x^\mu(t),\ z^i(t) \right)$$

$$\frac{d}{dt} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{z}^j \frac{\partial}{\partial z^j} \quad \text{, with} \quad \dot{z}^i + \Gamma^i_{\mu j} \dot{x}^\mu z^j = 0$$

Therefore, $\frac{d}{dt}$ becomes $\dot{x}^\mu \overbrace{\left(\partial_\mu - \Gamma^i_{\mu j} z^j \frac{\partial}{\partial z^i} \right)}^{D_\mu}$

$$\equiv \dot{x}^\mu D_\mu$$

$$\boxed{D_\mu = \partial_\mu - \Gamma^i_{\mu j} z^j \frac{\partial}{\partial z^i}}$$

D_μ : covariant derivative (along $\frac{\partial}{\partial x^\mu}$)

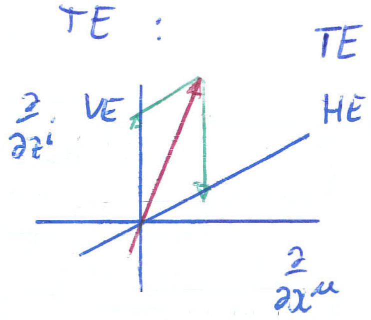
(alternative notation : ∇_μ)

We get a splitting (+) at $x \in U$ of TE :

$$T_xE = V_xE \oplus H_xE$$

vertical $\left(\frac{\partial}{\partial z^i} \right.$, horizontal $\left. D_\mu \right)$

(+) not "God-given": you need Γ



$\{ H_xE \}$: horizontal distribution in TE

③ One can formulate the above in terms of differential forms

$\sim \in T^*E$

Define $\omega = (\omega^i := dz^i + \Gamma^i_{\mu j} z^j dx^\mu)$

pulled back to T^*M via $z^i = z^i(x)$

ω is uniquely defined by the requirements:

$$(\diamond) \quad \omega^i(D_\mu) = 0$$
$$\omega^i(\partial_{z^j}) = \delta^i_j$$

and conversely, $(\diamond)$ determines D_μ

$\Gamma^i{}_j := \Gamma^i_{\mu j}\, dx^\mu$ matrix valued 1-form

Define the <u>total covariant derivative</u>

$$\nabla s = e_i \otimes dz^i + e_i \otimes \Gamma^i{}_j z^j \quad (\text{at } x)$$

This is the <u>pull-back</u> of $\nabla e_i = \Gamma^j_i e_j$

$$\nabla Z = e_i \otimes \omega^i \qquad (Z = e_i z^i \in \pi^{-1}(U))$$

Z horizontal: $\nabla Z = 0 \quad e_i \omega^i(Z) = 0$

④ Axiomatise: $\nabla_X : \Gamma(E) \to \Gamma(E)$

1. $\nabla_X(s + s') = \nabla_X s + \nabla_X s'$
2. $\nabla_{X+X'}(s) = \nabla_X s + \nabla_{X'} s$
3. $\nabla_X(fs) = \underset{df(X)}{X(f)}\, s + f \nabla_X s$ (Leibniz)
4. $\nabla_{fX} s = f \nabla_X s$ tensoriality in X

For the total covariant derivative: $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Lambda^1(M) = \Gamma(E \otimes T^*M)$

$\Gamma(E) \equiv C^\infty(E)$

1. $\nabla(s + s') = \nabla s + \nabla s'$
2. $\nabla f s = df\, s + f \nabla s$

Then $\nabla s = \nabla_{\frac{\partial}{\partial x^\mu}} s \otimes dx^\mu$

$$\nabla_X s = (\nabla s, X)$$

Extend to p-form valued sections of E.

$$\nabla(s \otimes \theta) = \nabla s \wedge \theta + s \otimes d\theta$$

$s \in C^\infty(E)$, $\theta \in C^\infty(\Lambda^p(M)) \equiv \Lambda^p(M)$

$$\nabla : C^\infty(E \otimes \Lambda^p(M)) \to C^\infty(E \otimes \Lambda^{p+1}(M))$$

⑤ **Change of frame**

$$e'_j = e_i \Phi^{-1}_{ij} \qquad \underbrace{e'}_{} = \underbrace{e}_{} \, \Phi^{-1}$$

Φ^{-1}_{ij} := (ij)-entry of Φ^{-1}

$$z'^i = \Phi_{ij} z^j$$

$$s(x) = e_i z^i = e_i \underbrace{\Phi^{-1}_{ik} \Phi_{kj}}_{\delta_{ij}} z^j = e'_k z'^k$$

$$\nabla(e'_j) = \nabla(e_i) \otimes \Phi^{-1}_{ij} + e_i \otimes d\Phi^{-1}_{ij} = e_i \Gamma'^i_j = e'_i \Gamma'^i_j$$

with

$$(\star) \quad \boxed{\Gamma'^i_j = \Phi_{ik} \Gamma^k_l \Phi^{-1}_{lj} + \Phi_{ik}\, d\Phi^{-1}_{kj}}$$

let us check this ⟹

$$e'_i \Gamma'^{i}_{j} \equiv e'_h \Gamma'^{h}_{j}$$

$$\boxed{\nabla e_i = \Gamma^j_i e_j}$$

$$\begin{aligned}
\nabla e'_j &= \nabla e_i\, \Phi^{-1}_{ij} + e_i\, d\Phi^{-1}_{ij} \\
&= \Gamma^k_i e_k\, \Phi^{-1}_{ij} + e_i\, d\Phi^{-1}_{ij} \\
&= \Gamma^k_i e'_h \Phi_{hk} \Phi^{-1}_{ij} + e'_h \Phi_{hi}\, d\Phi^{-1}_{ij}
\end{aligned}$$

$$e'_j = e_i \Phi^{-1}_{ij}$$

$$e'_j \Phi_{jk} = e_i \Phi^{-1}_{ij} \cdot \Phi_{jk} = e_i \delta_{ik} = e_k$$

$$= \underbrace{\left[\Phi_{hk} \Gamma^k_i \Phi^{-1}_{ij} + \Phi_{hi}\, d\Phi^{-1}_{ij}\right]}_{=\,\Gamma'^{h}_{j}} e'_h$$

$$\{e_k = e'_h \Phi_{hk}\}$$

$$\Gamma'^{h}_{j} e'_h \equiv e'_h \Gamma'^{h}_{j}$$

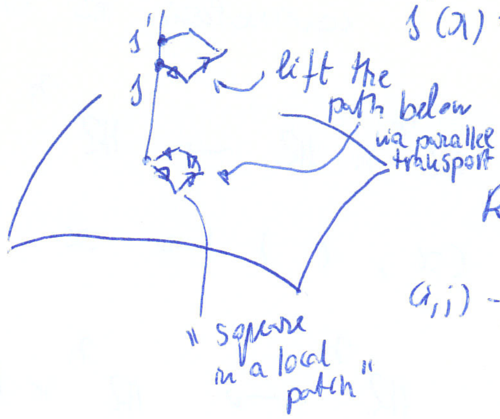
"gauge field transformation"

Therefore (A) ensures independence of ∇s from change of frame

Curvature of ∇

via ① (Sketch)

(lifting curves on M via parallel transport: closed curves below do not close upstairs, discrepancy being measured via a "holonomy matrix",



$s'(a) = H\, s(a)$ ↑ holonomy matrix

$R_{ij} := \frac{d}{d\tau} H_{ij}(\tau)\Big|_{\tau=0}$

(i,j)-plane

whose infinitesimalization yields the curvature operator, rapidly obtained by the following other methods

② As an obstruction to integrability of the horizontal distribution (indeed, one gets a vertical vector)

$$\boxed{[D_\mu, D_\nu] = -R^i_{\ j\mu\nu}\, z^j \frac{\partial}{\partial z^i}}$$

$R_{\mu\nu} = (R^i_{\ j\mu\nu})$ curvature operator

One computes

$$\boxed{R^i_{\ j\mu\nu} = \partial_\mu \Gamma^i_{\ \nu j} - \partial_\nu \Gamma^i_{\ \mu j} + \Gamma^i_{\ \mu k}\Gamma^k_{\ \nu j} - \Gamma^i_{\ \nu k}\Gamma^k_{\ \mu j}}$$

(cf. the calculation of the Riemann tensor)

③ R becomes a matrix-valued 2-form

$$\boxed{R^i_{\ j} = d\Gamma^i_{\ j} + \Gamma^i_{\ k}\wedge\Gamma^k_{\ j} = \frac{1}{2} R^i_{\ j\mu\nu}\, dx^\mu \wedge dx^\nu}$$

customary... notice: no dz terms

Also:

$$\boxed{R^i_{\ j} z^j = d\omega^i + \Gamma^i_{\ j}\wedge\omega^j}$$

~ "Cartan 1st structure equation"

covariant differential of ω^i

dz^i terms drop out in $R^i_{\ j}$

④

$$R(X,Y)s := \{[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\} s$$

$$= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

$$R(\partial_\mu, \partial_\nu)(e_i) = e_j R^j{}_{i\mu\nu}$$

local frame

Properties:

1. multilinearity $\quad R(X+X', Y) = R(X,Y) + R(X',Y)$
2. Antisymmetry $\quad R(X,Y) = -R(Y,X)$
3. Tensoriality

scalar function

$$R(fX,Y)\cdot s = R(X,fY)s = R(X,Y)(fs)$$

$$= f R(X,Y) s$$

Total curvature (curvature "tout court")

$$R\cdot s = \nabla^2 s = \nabla(\nabla s)$$ (2-form valued linear map from E to E)

Compute (& compare)

$s = e_j z^j$

$$\boxed{R s = \nabla(\nabla s) = \nabla(e_j \otimes \Gamma^j_i z^i + e_j \otimes dz^j)}$$

$$= \underbrace{e_k \otimes \Gamma^k_j}_{\nabla(e_j)} \wedge \Gamma^j_i z^i + e_k \otimes [d\Gamma^k_i z^i - \Gamma^k_i \wedge dz^i]$$

$$+ e_k \otimes \Gamma^k_j \wedge dz^j + 0 \quad (d^2 = 0)$$

cancel out

$$= e_k \otimes (\Gamma^k_j \wedge \Gamma^j_i + d\Gamma^k_i) z^i$$

$$= e_k \otimes R^k{}_i z^i \qquad R = \frac{1}{2} R(\partial_\mu, \partial_\nu) dx^\mu \wedge dx^\nu$$

(5) Upon frame change, we have

$$R'^{i}{}_{j} = \Phi^{i}{}_{k} R^{k}{}_{h} (\Phi^{-1})^{h}{}_{j} \qquad \text{(tensorial)}$$

Therefore R's is intrinsically defined (invariant under change of frame)

$\ast\ast\ast$ <u>geometrical significance of R</u>:

obstruction to finding <u>covariantly constant</u> (locally flat) <u>frames</u>:

given (e_i), try & find $e'_i = e_j \Phi^{-1}_{ji}$ locally flat:

(∗) $\nabla e'_i = 0$: we get

$$\phi \Gamma \phi^{-1} + \phi \, d\phi^{-1} = 0$$

$$\Rightarrow \quad \Gamma = - d\phi^{-1} . \phi \qquad \text{("pure gauge")}$$

but (∗) implies $\nabla^2 e'_i = 0 \quad \Rightarrow \quad R = 0$

also compute

$R = d\Gamma + \Gamma \wedge \Gamma$ (Cartan)

$$d\Gamma = + d\phi^{-1} \wedge d\phi$$

$$\Gamma \wedge \Gamma = d\phi^{-1} \phi \, d\phi^{-1} \phi$$
$$= - d\phi^{-1} d\phi \, \phi^{-1} \phi$$
$$= - d\phi^{-1} \wedge d\phi$$

$0 = d(\phi \phi^{-1})$
$= d\phi \, \phi^{-1} + \phi \, d\phi^{-1}$
$\phi \, d\phi^{-1} = - d\phi \, \phi^{-1}$
($\wedge$ omitted)

hence $R = 0$, of course...

Let us provide some details

$$R(X,Y)\, e_j = \Omega^i{}_j(X,Y)\, e_i$$

$\nabla_X e_j = \omega^i{}_j(X)\, e_i$

$\omega = (\omega^i{}_j) \sim \omega \sim \Gamma$

$\Omega = (\Omega^i{}_j)$ $\sim R$

We wish to check that

$$(\ast) \quad \boxed{\Omega = d\omega + \omega\wedge\omega} \qquad (\text{Cartan})$$

$$\underbrace{R(X,Y)\, e_j}_{\Omega^i{}_j(X,Y)e_i} = \left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right) e_j$$

$$= \nabla_X\left[\omega^i{}_j(Y)e_i\right] - \nabla_Y\left[\omega^i{}_j(X)e_i\right] - \omega^i{}_j([X,Y])e_i$$

$$= X(\omega^i{}_j(Y))\, e_i + \omega^k{}_j(Y)\,\omega^i{}_k(X)\, e_i$$

$$- Y(\omega^i{}_j(X))\, e_i - \omega^k{}_j(X)\,\omega^i{}_k(Y)\, e_i - \omega^i{}_j([X,Y])\, e_i$$

use $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$

$$= \left\{ d\omega^i{}_j(X,Y) + \cancel{\omega^i{}_j([X,Y])} + \omega^k{}_j\wedge\omega^i{}_k(X,Y) - \cancel{\omega^i{}_j([X,Y])} \right\} e_i$$

$$= \left(d\omega^i{}_j + \omega^k{}_j\wedge\omega^i{}_k\right) e_i$$

and this is $(\ast)$.

☆ Bianchi identity $\quad (\ast) \Rightarrow \Big\{ d\Omega = \overset{0}{d^2\omega} + d\omega\wedge\omega - \omega\wedge d\omega$

$$= (\Omega - \omega\wedge\omega)\wedge\omega - \omega\wedge(\Omega - \omega\wedge\omega) = \Omega\wedge\omega - \omega\wedge\Omega \Big\}$$

i.e.

$$\boxed{d\Omega = \Omega\wedge\omega - \omega\wedge\Omega}$$