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Lecture LXV

CONNECTIONS ON PRINCIPAL BUNDLES

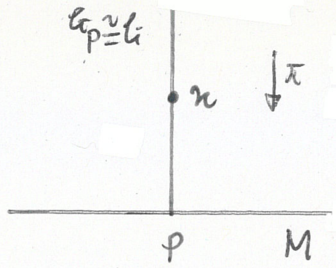
★ Connections on principal bundles

$\pi : P \xrightarrow{G} M$ or briefly $P(M, G)$ (or P)

principal bundle with

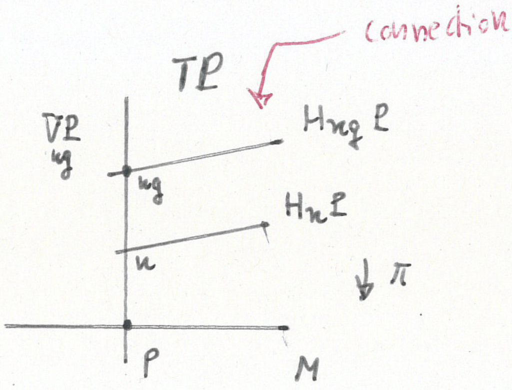
structure group G (Lie group)

$G_p \cong G$ fibre at $p \in M$



vertical subspace

$V_x P$: subspace of $T_x P$ tangent to G_p at x



Let $A \in \mathfrak{g}$ (the Lie algebra of $\mathfrak{g}$)

(for simplicity, think of a matrix group)

Consider the right action

$$R_{\exp(tA)}\, x = x \exp tA$$

giving rise to a curve in P. Clearly, since

$$\boxed{\pi(x \underbrace{\exp tA}_{g}) = \pi(x) = p}$$, this is a curve in G_p.

Its velocity field at x, $A^{\#}_x$ is defined (as usual)

via
$$\boxed{(A^{\#} f)(x) := \frac{d}{dt} f(x \cdot \exp tA)\Big|_{t=0}}$$

$f \in C^\infty(P)$

$A^{\#}_x \in V_x P$. Actually one gets a vector field $A^{\#}$

(fundamental vector field associated to $A \in \mathfrak{g}$)

symbolically $\quad \# : \mathfrak{g} \to V(P) \subset \mathfrak{X}(P)$ (linear map)

$\qquad A \mapsto A^{\#}$ vertical v. fields

Clearly $\pi_* X = 0 \quad \forall X \in VP$

$(X_u \in V_u P \quad \forall u \in P)$

and $[A^\#, B^\#] = [A, B]^\#$

(so # becomes a Lie algebra homomorphism)

A connection is a (smooth) choice of a complement to $V_u P$ at each $u \in P$ (horizontal subspace $H_u P$) such that:

(i) $T_u P = H_u P \oplus V_u P$ (this formalises)

(ii) $H_{ug} P = (R_g)_* H_u P \quad \forall u \in P, \forall g \in G$

"covariance"

Any $X \in \mathfrak{X}(P)$ can be decomposed into a horizontal & a vertical part (obvious notation)

$$X = X^V + X^H$$

Operationally, the splitting induced by the connection (à la Ehresmann) can be implemented via a $\underline{g}$-valued one-form $\omega \in \underline{g} \otimes T^*P$ (connection 1-form)

A connection 1-form $\omega \in \underline{g} \otimes T^*P$ is an "oblique" projection of $T_\cdot P$ onto $V_\cdot P \cong \underline{g}$, satisfying:

(i) $\omega(A^\#) = A \quad \forall A \in \underline{g}$

(ii) $R_g^* \omega = Ad_{g^{-1}} \omega$

i.e for $X \in T_u P$: $R^*_g \omega_{ug}(X) = \omega_{ug}(R_{g*} X) = g^{-1} \overset{\underline{g}}{\downarrow}\omega_u(X) g$

matrix group notation

* The horizontal subspace $H_u P$ is then the kernel of ω :

$$\boxed{H_u P := \{ X \in T_u P \,/\, \omega(X) = 0 \}} \quad (\diamond)$$

Let us check that the above position is well-defined. So let $X \in H_u P$ (defined via $(\diamond)$). Then

$$\omega((R_g)_* X) = R_g^* \omega(X) = g^{-1} \underbrace{\omega(X)}_{=0} g = 0$$

The invertibility of $(R_g)_*$ then leads to the desired conclusion.

* Local description

Let $\{\mathcal{U}_i\}$ be an open covering of M, let σ_i be a local section $\sigma_i : \mathcal{U}_i \to P$ $(\pi \circ \sigma_i = id_M)$

Let A_i ($\mathfrak{g}$-valued form on $\mathcal{U}_i$) defined as follows:

$$(\diamond) \quad A_i := \sigma_i^* \omega \in \mathfrak{g} \otimes \Lambda^1(\mathcal{U}_i)$$

We shall show that:

(1) Conversely, given A_i and σ_i, one can produce $\omega_i \in \mathfrak{g} \otimes \Lambda^1(\pi^{-1}(\mathcal{U}_i))$ such that

* connection form

$$\boxed{A_i = \sigma_i^* \omega_i}$$

Notice that everything takes place on a fixed $\mathcal{U}_i$

(2) If the A_i fulfil a certain "Christoffel-like" condition on non void overlappings $\mathcal{U}_i \cap \mathcal{U}_j$, then one can find ω (global) such that $(\diamond)$ holds.

Ad ① : Define ω_i as follows:

Maurer-Cartan

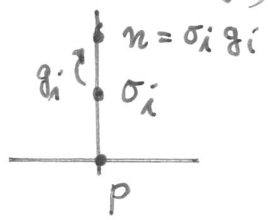
$$\omega_i = g_i^{-1}\,\pi^* A_i\, g_i + \underbrace{g_i^{-1}\, d\, g_i}$$

$d = d_P$ (differential on P)

!! we are using the canonical local trivialisation defined by the local section σ_i:

$$u = \sigma_i(p)\, g_i \quad \rightsquigarrow \quad u = (p, g_i)$$

↗ this defines g_i



Let us check that $\boxed{\sigma_i^*\omega_i = A_i}$

$$\sigma_i^*\omega_i(\overset{T_pM}{X}) = \omega_i(\sigma_{i*}X) = \pi^* A_i[(\sigma_i)_* X] + dg_i((\sigma_{i*})X)$$

$\uparrow g_i = 1$

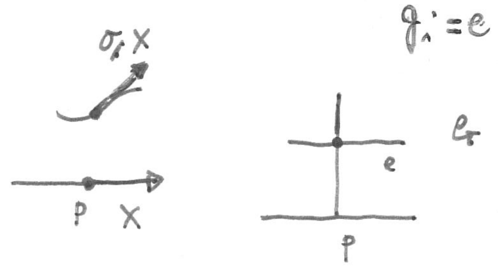
! $(\sigma_i)_* X \in T_{\sigma_i}P$ so $g_i = e$ at $u = \sigma_i$

$$\| \qquad (\sigma_{i*}X)(\overset{e}{\overset{\|}{g_i}})$$

$$A_i(\underbrace{\pi_*\sigma_{i*}}_{=\,\mathrm{id}} X) + dg_i(\sigma_{i*}X)$$

$$\| \; 0 \qquad g_i = e$$

$$= A_i(X)$$



In order to complete ①, we have to show that ω_i is indeed a connection form

(i) Take $A \in \mathfrak{g}$ and $X = A^{\#} \in V_\bullet P$

We have $\pi_* X = 0$. Subsequently

$$\omega_i(A^{\#}) = g_i^{-1}\, d g_i(A^{\#}) = \underset{\underset{P}{\uparrow}}{g_i^{-1}(x)} \left.\frac{d g_i(x\cdot \exp tA)}{dt}\right|_{t=0}$$

$$= g_i(x)^{-1} g_i(x) \left.\frac{d \exp(tA)}{dt}\right|_{t=0} = A$$

(ii) Let $X \in T_x P$, $h \in G$. Then

$$R_h^* \omega_i(X) = \omega_i((R_h)_* X) = \overset{h^{-1}g_i^{-1}(x)}{g_i^{-1}(xh)}\, A_i(\overset{\pi_* X}{\pi_* R_{h*} X})\, \overset{g_i(x)\cdot h}{g_i(xh)}$$
$$+ \underbrace{g_i^{-1}(xh)\, d g_i(xh)\, (R_{h*} X)}_{(\blacklozenge)}$$

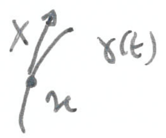
Now $g_i(xh) = g_i(x) h \Rightarrow g_i^{-1}(xh) = (g_i(x) h)^{-1}$
$= h^{-1} g_i^{-1}(x)$

Also $\pi_* R_{h*} X = \pi_* X$ (since $\pi R_h = \pi$)

Moreover: $(\blacklozenge) = g_i^{-1}(xh)\, d\, g_i(xh) (R_{h*} X) =$

$$= g_i^{-1}(xh) \cdot \left.\frac{d}{dt} g_i(\gamma(t)\cdot h)\right|_{t=0}$$

with $\gamma = \gamma(t)$ a curve through $x = \gamma(0)$ with velocity X thereat



$$= h^{-1} g_i^{-1}(x) \left.\frac{d}{dt} g(\gamma(t))\right|_{t=0} \cdot h$$

$$= h^{-1}\, g_i^{-1}(x)\, d g_i(x)\, (X) \cdot h$$

Collecting everything we get

$$\boxed{R_h^* \omega_i(X) = h^{-1} \cdot \omega_i(X) \cdot h}$$

and this completes step 1

Ad ②

The previous construction is global so long as the various forms $\{\omega_i\}$ agree on overlappings:

$$\omega_i = \omega_j \quad \text{on} \quad \mathcal{U}_i \cap \mathcal{U}_j \quad (\neq \emptyset)$$

satisfying

Let $\sigma_{i,j}$ two local sections on $\mathcal{U}_{i,j}$

If $p \in \mathcal{U}_i \cap \mathcal{U}_j$, $X \in T_pM$, we have

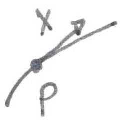
$$\boxed{\sigma_{j*}X = R_{t_{ij}*}(\sigma_{i*}X) + (t_{ij}^{-1}\,d t_{ij}(X))^{\#}}$$

Technical lemma

$(t_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \to G$ transition function

$\sigma_j(p) = \sigma_i(p)\, t_{ij}(p))$ (*)

Let $\gamma : [0,1] \to M$ s.t. $\gamma(0) = p$, $\dot\gamma(0) = X$



$$\sigma_{j*}X = \frac{d}{dt}\sigma_j(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\left[\sigma_i(t)\, t_{ij}(t)\right]\Big|_{t=0}$$

$$= \left[\overbrace{\left(\frac{d}{dt}\sigma_i(t)\right)\cdot t_{ij}(p)}^{R_{t_{ij}}(\sigma_{i*}X)} + \sigma_i(p)\frac{d}{dt}t_{ij}(t)\right]\Big|_{t=0}$$

recall: G matrix group $(R_g)_* X = Xg$

$$= R_{t_{ij}}(\sigma_{i*}X) + \sigma_j(p)\, t_{ij}^{-1}(p)\frac{d}{dt}t_{ij}(t)\Big|_{t=0}$$ (✝) (use (*))

Now $t_{ij}(p)^{-1}\, d t_{ij}(X) = t_{ij}(p)^{-1}\frac{d}{dt}t_{ij}(t)\Big|_{t=0} =$

$$= \frac{d}{dt}\left[t_{ij}(p)^{-1}t_{ij}(t)\right]_{t=0} \in T_e(G) \cong \mathfrak{g}$$

$t_{ij}^{-1}(p)\, t_{ij}(\gamma(0)) = e$

whence (✝) represents $(t_{ij}^{-1}\, d t_{ij}(X))^{\#}$ at p

Then compute:

$$\sigma_j^* \omega(X) = R^*_{t_{ij}} \omega(\sigma_{i*} X) + t_{ij}^{-1} d t_{ij}(X)$$

$$= t_{ij}^{-1} \omega(\sigma_{i*} X) t_{ij} + t_{ij}^{-1} d t_{ij}(X)$$

Therefore, the compatibility condition we were looking for reads

Maurer-Cartan

$$\boxed{A_j = t_{ij}^{-1} \cdot A_i t_{ij} + t_{ij}^{-1} d t_{ij}}$$

$\{A_i\}$ local gauge potentials

"non-tensorial part"

$A_i = \sigma_i^* \omega$

in general, given two local sections σ_1 and σ_2 over U with $\sigma_2(p) = \sigma_1(p) \cdot g(p)$, then

$$\boxed{A_2 = g^{-1} A_1 g + g^{-1} d g}$$

** gauge transformation

or, in components:

$$\boxed{(A_2)_\mu = g^{-1} (A_1)_\mu g + g^{-1} \partial_\mu g}$$

Step ② is completed as well. □

Aside

Generalized commutators

$\mathfrak{g}$: Lie algebra

$$[T_a, T_b] = f_{ab}^{\ \ c} T_c$$

structure constants

$$\xi = \xi^a \otimes T^a$$

$$[\overset{p}{\xi}, \overset{q}{\eta}] := \xi \wedge \eta - (-1)^{pq} \eta \wedge \xi$$

$$= T_a T_b \, \xi^a \wedge \eta^b - (-1)^{pq} T_b T_a \, \eta^b \wedge \xi^a$$

$$\xi^a \wedge \eta^b = (-1)^{pq} \eta^b \wedge \xi^a$$

$$= [T_a, T_b] \, \xi^a \wedge \eta^b = f_{ab}^{\ \ c} T_c \, \xi^a \wedge \eta^b$$

If $\xi = \eta$ we have

$$[\xi, \xi] = 2 \cdot \xi \wedge \xi = f_{ab}^{\ \ c} T_c \, \xi^a \wedge \xi^b$$

consistently with the preceding treatment