** Curvature

In general: If $\phi \in \Lambda^r(P) \otimes V$, given a **connection** (form) ω,

$$\left(\phi = \phi^\alpha \otimes e_\alpha\right)$$

vector space

$\phi^\alpha \in \Lambda^r(P)$; basis of V

one can define the covariant derivative $D\phi$ (or $\nabla\phi$)

as follows:

horizontal components

$$\boxed{D\phi\,(X_1 \ldots X_{r+1}) = \overset{d_P}{\overset{\shortparallel}{d}}\phi\,(X_1^H \ldots X_{r+1}^H)}$$

(with $d\phi = d\phi^\alpha \otimes e_\alpha$)

* Curvature 2-form $\boxed{\Omega \in \Lambda^2(P) \otimes \mathfrak{g}}$

$$\boxed{\Omega := D\omega}$$

(ω : connection form)

One has:

$$\boxed{R_a^*\,\Omega = a^{-1}\,\Omega\, a} \qquad a \in G$$

Proof. $(R_{a*} X)^H = R_{a*} X^H$

Indeed, first notice that $\omega(R_{a*}X^H) = R_a^*\omega(X^H) = 0$,

this meaning that $(R_a)_* X^H$ is horizontal.

Then $\underbrace{(R_{a*}X)^H}_{\text{horizontal}} = (R_{a*}X^H + R_{a*}X^V)^H = (\overbrace{R_{a*}X^H}^{R_{a*}X^H})^H + \underbrace{(R_{a*}X^V)^H}_{\substack{\text{vertical} \\ \shortparallel \\ 0}} =$

$$= R_{a*}X^H + 0 = R_{a*}X^H$$

Also, $d_P R_a^* = R_a^* d_P$

Then

$$\boxed{(R_a^* \Omega)(X,Y)} = \Omega(R_{a*}X, R_{a*}Y) = d\omega((R_{a*}X)^H, (R_{a*}Y)^H)$$
$$= d\omega(R_{a*}X^H, R_{a*}Y^H) = R_a^* d\omega(X^H, Y^H)$$
$$= d\, R_a^*\omega(X^H, Y^H) = d(a^{-1}\omega a)(X^H, Y^H)$$

$(da = 0)$ $\qquad = a^{-1} d\omega(X^H, Y^H)\, a$

a is constant... $\qquad = \boxed{a^{-1}\, \Omega(X,Y)\, a}$ $\square$

<u>Proposition</u> $X \in H.P$, $Y \in V.P \Rightarrow [X,Y] \in H.P$

i.e. [hor, vert] = hor

Let $g = g(t)$ the 1-parameter group generated by Y

One has

$$[Y,X] = \lim_{t \to 0} \frac{(R_{g(t)*}X - X)}{t}$$

But $R_{g*} H_u P = H_{ug} P$ (by definition)

So $R_{g(t)*}X$ is horizontal, whence $[X,Y] = -[Y,X]$ is horizontal too.

★ **Theorem** Let $X, Y \in T_p P$. We have the Cartan structure equation

$$(\blacklozenge) \quad \boxed{\Omega(X,Y) = \underset{\underset{d_P}{\wr}}{d}\omega(X,Y) + [\omega(X), \omega(Y)]}$$

or equivalently

$$(\blacklozenge') \quad \boxed{\Omega = d\omega + \frac{1}{2}[\omega, \omega] = d\omega + \omega \wedge \omega}$$

Let us check compatibility of $(\blacklozenge)$ and $(\blacklozenge')$ first:

$$[\omega, \omega](X,Y) = [T_a, T_b]\, \omega^a \wedge \omega^b (X,Y) \qquad (-1)^{pq} = (-1)^{1\cdot 1} = -1$$

$$\underset{2\,\omega\wedge\omega\,(X,Y)}{\parallel} \quad = [T_a, T_b]\left\{\omega^a(X)\,\omega^b(Y) - \omega^a(Y)\,\omega^b(X)\right\}$$

$$= [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)]$$

$$= 2\,[\omega(X), \omega(Y)]$$

We then split the proof into several cases:

① X, Y horizontal

$\omega(X) = \omega(Y) = 0$ and $\Omega(X,Y) = d\omega(X,Y)$ ✓

② X horizontal, Y vertical (and it is enough to take a fundamental vector field)

$Y^H = 0 \Rightarrow \Omega(X,Y) = 0$. Also $\omega(X) = 0$

We have to check that $d\omega(X,Y) = 0$. But

$$d\omega(X,Y) = X\omega(Y) - Y\underbrace{\omega(X)}_{=0} - \omega([X,Y])$$

$$= X\,\omega(Y) - \omega(\underbrace{[X,Y]}_{\text{horizontal}}) = X\,\underbrace{\omega(Y)}_{=A \in \mathfrak{g}} = 0$$

constant

$\omega(\text{horizontal}) = 0$

③ X, Y vertical (enough to take fundamental vector fields)

$$\Omega(X,Y) = 0$$

$$d\omega(X,Y) = \underbrace{X\,\omega(Y)}_{\substack{\text{constant} \\ = 0}} - \underbrace{Y\,\omega(X)}_{\substack{\text{constant} \\ = 0}} - \omega([X,Y]) = -\omega(\underbrace{[X,Y]}_{\text{vertical}})$$

put $A^{\#} = [X,Y]$. If $B^{\#} = X$, $C^{\#} = Y$, we have f.v.f.

$$[\omega(X), \omega(Y)] = [B, C] = A$$

(recall that $[A,B]^{\#} = [A^{\#}, B^{\#}]$)

Therefore $d\omega(X,Y) + [\omega(X), \omega(Y)] = 0$ ✓

Cartan's formula is completely proven. □

✱ geometrical significance of Ω

Let X, Y be two horizontal vector fields

Then $\boxed{\Omega(X,Y) = -\omega([X,Y])}$ (✱)

If $\Omega(X,Y) \neq 0$, then $[X,Y]$ is not horizontal and conversely:

The form Ω measures the obstruction to integrability of the horizontal distribution (The vertical distribution, by contrast, is always integrable by its very definition)

Let us check (✱). We have $\omega(X) = \omega(Y) = 0$

$$\Omega(X,Y) = d\omega(X,Y) + [\underbrace{\omega(X)}_{=0}, \underbrace{\omega(Y)}_{=0}]$$

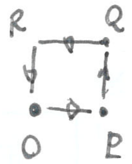
$$= d\omega(X,Y) = X\underbrace{\omega(Y)}_{=0} - Y\underbrace{\omega(X)}_{=0} - \omega([X,Y])$$

$$= -\omega([X,Y])$$

Let us interpret Ω in terms of parallel transport

Work locally in a coordinate patch, with local coordinates $\{x^\mu\}$ take $V = \frac{\partial}{\partial x^1}$ $W = \frac{\partial}{\partial x^2}$ to fix ideas

consider the "small" parallelogram γ:



$O: \{0\}$ $P = (\varepsilon, 0 \ldots 0)$

$Q = (\varepsilon, \delta, 0 - 0)$

$R = (0, \delta, 0 - 0)$

ε, δ suff. small

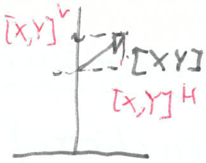
Lift γ horizontally to $\tilde{\gamma}$.

Let X s.t. $\pi_* X = \varepsilon V$

$\underbrace{Y}_{\text{horizontal}}$ " $\pi_* Y = \delta W$

Then $\quad \pi_*\big(\underset{= [\pi_* X,\, \pi_* Y]}{[X,Y]}\big) = \varepsilon\delta\, \underbrace{[V,W]}_{=0}$

(π-related objects)



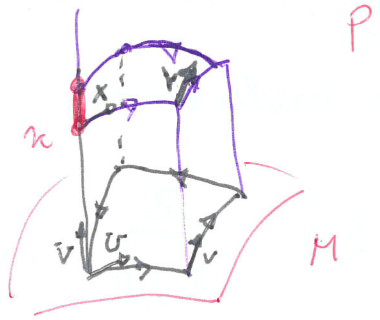
$\Rightarrow \quad [X,Y] = [X,Y]^V \quad$ i.e. $[X,Y]$ is <u>vertical</u>

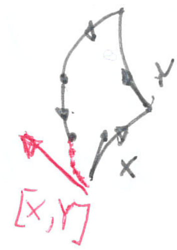
Then, from $\Big[\, \Omega(\overset{=X^h}{X}, \overset{=Y^h}{Y}) = -\omega([X,Y]) = -\omega([X,Y]^V)$

ω is a <u>projection</u>

$= -[X,Y]^V = -[X,Y] = -[X^h, Y^h] \,\Big]$

we get that the <u>vertical</u> vector $\boxed{\Omega(X^h, Y^h)}$ measures lack of commutativity of the horizontal lifts X^h, Y^h (cf. H. Baum, cover!)





In this context we still have Bianchi:

$$\boxed{d\Omega = \Omega \wedge \omega - \omega \wedge \Omega}$$

Equivalent form: $\boxed{D\Omega = 0}$ indeed

$$\Big[\, D\Omega(X,Y,Z) = d\Omega(X^H, Y^H, Z^H) =$$

$$(\Omega \wedge \omega)(X^H, Y^H, Z^H) - (\omega \wedge \Omega)(X^H, Y^H, Z^H) = 0 \,\Big]$$

since $\quad \omega(\mathrm{Hor}) = 0$