★ Basic fact: $\boxed{d(\operatorname{Tr}\Omega^i) = 0}$

$\Omega^i = \underbrace{\Omega \wedge \Omega \wedge \dots \Omega}_{i \text{ times}}$

$(\Rightarrow \quad d\, P(\Omega) = 0$

$\forall P$ invariant polynomial$)$

Proof

$d \operatorname{Tr} \Omega^i = \operatorname{Tr} d\Omega^i =$

$= \operatorname{Tr}(d\Omega \wedge \Omega^{i-1} + \Omega \wedge d\Omega \wedge \Omega^{i-2} + \dots + \Omega^{i-1} \wedge d\Omega)$

Bianchi   Bianchi

$= \operatorname{Tr}((\Omega \wedge \omega - \omega \wedge \Omega) \wedge \Omega^{i-1} + \Omega \wedge (\Omega \wedge \omega - \omega \wedge \Omega) \wedge \Omega^{i-2} + \dots)$

$= \operatorname{Tr}(\Omega^i \wedge \omega - \omega \wedge \Omega^i + \dots)$

$= \operatorname{Tr}(\omega \wedge \Omega^i - \omega \wedge \Omega^i + \dots$

$= \dots = 0$

$\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$

$\operatorname{Tr}(XYZ) = \operatorname{Tr}(YZX) = \operatorname{Tr}(ZXY)$

Special case $i = 2$

$d \operatorname{Tr} \Omega^2 = \operatorname{Tr} d\Omega^2 = \operatorname{Tr}(d\Omega \wedge \Omega + \Omega \wedge d\Omega)$

$= \operatorname{Tr}((\Omega \wedge \omega - \omega \wedge \Omega) \wedge \Omega - \Omega \wedge (\Omega \wedge \omega - \omega \wedge \Omega))$

$= \operatorname{Tr}(\Omega \wedge \omega \wedge \Omega - \omega \wedge \Omega \wedge \Omega - \dots)$

$= \operatorname{Tr}(\Omega \wedge \Omega \wedge \omega - \omega \wedge \Omega \wedge \Omega -$

$= \operatorname{Tr}(\omega \wedge \Omega^2 - \omega \wedge \Omega^2 - \dots$

$= \dots = 0$

It is instructive to work out this very example in full detail (i.e. in components), see next page

CHARACTERISTIC CLASSES OF VECTOR BUNDLES

Lecture LXVII

Lectures on DIFFERENTIAL GEOMETRY & TOPOLOGY

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★ Explicit computation (sample)

★ **Bianchi:** $d\Omega = \Omega\wedge\omega - \omega\wedge\Omega$

in components

$$\boxed{d\Omega^i{}_j = \Omega^i{}_k\wedge\omega^k{}_j - \omega^i{}_k\wedge\Omega^k{}_j}$$ (Einstein...)

Let us compute: $\boxed{d\,\mathrm{Tr}(\Omega^2)} = \mathrm{Tr}\, d(\Omega\wedge\Omega)$

$$= \mathrm{Tr}(d\Omega\wedge\Omega - \underbrace{\Omega\wedge d\Omega}_{\diamond})$$

$$= \mathrm{Tr}(\underbrace{\Omega\wedge\omega\wedge\Omega - \omega\wedge\Omega\wedge\Omega}_{\clubsuit} - (\diamond))$$

we compute $\mathrm{Tr}(\clubsuit) = \mathrm{Tr}(\underbrace{\Omega^i{}_k\wedge\omega^k{}_j\wedge\Omega^j{}_l - \omega^i{}_k\wedge\Omega^k{}_j\wedge\Omega^j{}_l}_{(a^i{}_l)})$

$$= a^i{}_i = \Omega^i{}_k\wedge\omega^k{}_j\wedge\Omega^j{}_i - \omega^i{}_k\wedge\Omega^k{}_j\wedge\Omega^j{}_i$$

$$= \underbrace{\omega^k{}_j\wedge\Omega^i{}_k}_{\substack{\text{here let} \\ i\to j \\ j\to k \\ k\to i}}\wedge\Omega^j{}_i - \omega^i{}_k\wedge\Omega^k{}_j\wedge\Omega^j{}_i$$

recall $\omega^{(k)}\wedge\omega^{(h)} = (-1)^{kh}\,\omega^{(h)}\wedge\omega^{(k)}$

swap

$$= \omega^i{}_k\wedge\Omega^j{}_i\wedge\Omega^k{}_j - \omega^i{}_k\wedge\Omega^k{}_j\wedge\Omega^j{}_i$$

$$= \omega^i{}_k\wedge\Omega^j{}_i\wedge\Omega^k{}_j - \omega^i{}_k\wedge\Omega^j{}_i\wedge\Omega^k{}_j$$

$$= \boxed{0}$$

Similarly $\mathrm{Tr}(\diamond) = 0$ → Therefore $d\,\mathrm{Tr}(\Omega^2) = 0$

The general pattern should be clear.

Theorem The cohomology class $[f(\Omega)] \in H^{2k}_{dR}(M)$ is independent of the choice of the connection ∇.

invariant polynomial of degree k

$\Omega = d\omega + \omega\wedge\omega$ curvature form of a connection ∇ on $E \to M$

We shall give two proofs of this basic result, both instructive.

① Let ∇^0, ∇^1 two connections, with respective curvature forms Ω^0 and Ω^1

One forms the vector bundle

$$\pi \times id : E \times \mathbb{R} \to M \times \mathbb{R}$$

Define $\hat{\nabla}$ on $E\times\mathbb{R}$ as follows:

If $s \in \Gamma(E\times\mathbb{R})$ such that $s(x,t) = s(x)$

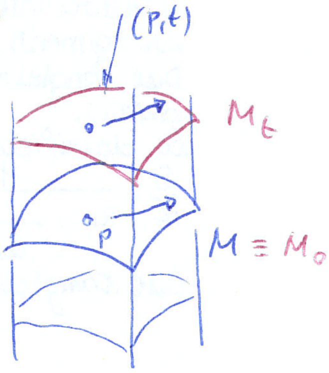
(i.e. s is independent of t, set

(a) $\hat{\nabla}_{\frac{\partial}{\partial t}} s = 0$

(and, for a generic section $\hat{s}(x,t) = f(x,t)s(x)$ and v. field $\alpha \frac{\partial}{\partial t}$

$$\hat{\nabla}_{\alpha\frac{\partial}{\partial t}} \hat{s} = \alpha \frac{\partial}{\partial t} f(x,t)\, s(x)$$

$\alpha = \alpha(x,t)$



(b) if $X \in T_{(p,t)}(\underbrace{M\times\{t\}}_{M_t})$, $p \in M$, $t \in \mathbb{R}$

$$\hat{\nabla}_X s = (1-t)\nabla^0_X s + t\nabla^1_X s$$

(remember that ∇ is tensorial in X)

(& extend by Leibniz rule and to any vector field Y, since the latter has necessarily the form

$$Y(x,t) = \alpha(x,t)\frac{\partial}{\partial t} + \underbrace{X(x,t)}_{= X^i(x,t)\frac{\partial}{\partial x^i}}$$

in a local chart

Consider the natural inclusion maps $\varepsilon = 0, 1$

$$i_\varepsilon : M \hookrightarrow M \times \mathbb{R}$$
$$x \longmapsto (x, \varepsilon)$$

We have $i_\varepsilon^* \tilde{\Omega} = \Omega^\varepsilon$

curvature of $\tilde{\nabla}$

But i_0 and i_1 are homotopic, thus, by homotopy invariance of de Rham Cohomology, we have

$$[f(\Omega^0)] = i_0^*[f(\tilde{\Omega})] = i_1^*[f(\tilde{\Omega})] = [f(\Omega^1)]$$

② [Following Nash & Sen '83 & 2011]

$P(x_1 \dots x_j)$ invariant polynomial of degree j

$\quad |||$ stemming from an invariant symmetric

$P(x \dots x) \equiv P_j(x)$ multilinear form in j variables

Let us again quickly prove that $P_j(F)$ is closed

here $F = dA + A \wedge A$
$\uparrow$
connection form

$$dP_j(F) = P(DF, F, \dots) + P(F, DF, F \dots) + P(F, \dots DF)$$

$$= j\, P(DF, F, \dots F)$$

D: covariant differential
i.e.: $Da = da + A \wedge a + a \wedge A$

but $DF = 0$ (Bianchi)

so $\boxed{dP_j(F) = 0}$

Now let us prove independence of the ensuing cohomology class from the choice of the connection

Let A', A be two connection, define the new connection (form)

$$A^t = tA' + (1-t)A = A + t\,(\overbrace{A'-A}^{a}) = A + ta$$

$$F^t = dA^t + A^t \wedge A^t = d(A+ta) + (A+ta) \wedge (A+ta)$$

$\quad ||$
curvature of A^t

$$= \underbrace{dA + A \wedge A}_{F} + t\,\underbrace{(da + A \wedge a + a \wedge A)}_{Da}$$

$(F^0 = F,\ F^1 = F')$

$$+ t^2\, a \wedge a$$

For subsequent use, notice that

$$(\clubsuit) \qquad \left.\frac{dF^t}{dt}\right|_{t=0} = Da$$

we can write

$$P_j(F') - P_j(F) = \int_0^1 \frac{d}{dt} \overbrace{P_j(F^t)}^{(*)} dt$$

we shall see in a minute that $(*)$ is exact for $0 \le t \le 1$: $\left[\frac{dP_j}{dt}(F^t) = d\,\theta(t) \right]$ for some $\theta(t)$

given this, we have

$$\boxed{P_j(F') - P_j(F)} = \int_0^1 d\,\theta(t) \cdot dt = d \overbrace{\int_0^1 \theta(t)\,dt}^{\overset{\eta}{\equiv}}$$

$$= d\,\eta$$

Now, it is enough to check exactness at $t=0$: it will then ensured for all t .. just replace $[0,1]$ with $[t,1]$ and A by A^t

$$\frac{d}{dt} P_j(F^t) = \frac{d}{dt} P(F^t, \ldots F^t) = P\left(\frac{dF^t}{dt}, F^t \ldots\right) + \ldots + P\left(F^t \ldots \frac{dF^t}{dt}\right)$$

But, by virtue of $(\diamond)$ $\boxed{\left.\frac{dF^t}{dt}\right|_{t=0} = Da}$

we have, at $t=0$

$$\boxed{\left.\frac{d}{dt} P_j(F^t)\right|_{t=0}} = P(Da, F - F) + P(F, Da, F -) \ldots + P(F - F, Da)$$

$$= j\,P(Da, F \ldots F)$$

$$= d\,(j\,P(a, F - F)) \quad \text{(Bianchi again)}$$

$$\equiv d\,\theta(0)$$

This achieves the conclusion. $\square$