Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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Lecture XLI.

SINGULAR HOMOLOGY

◇ singular homology

+ preliminaries: affine p-simplices

Let $\mathbb{R}^\infty = \bigoplus_{n=0}^{\infty} \mathbb{R}^{(n)} = \langle e_0, e_1, \dots \rangle$

$\mathbb{R}^{(n)}$: a copy of $\mathbb{R}$

$\equiv$ finite linear combinations of $e_i = (0, 0, \dots, 1, \dots 0)$, with the 1 in position i — "canonical basis"

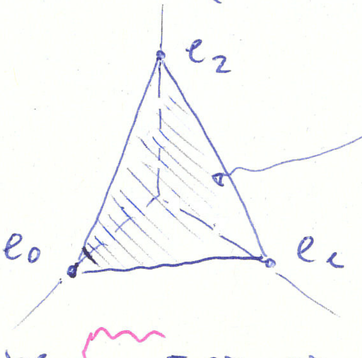
It will be used as a sort of general receptacle.

The standard p-simplex is, by definition

$$\boxed{\Delta_p = \left\{ x = \sum_{i=0}^{p} \lambda_i e_i \;/\; 0 \le \lambda_i \le 1 \,,\; \sum_{i=0}^{p} \lambda_i = 1 \right\}}$$

$\equiv \{$ convex combinations of $e_0, e_1, \dots, e_p \}$

The λ's are called barycentric coordinates



$p = 2$

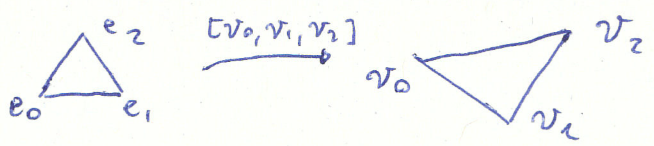
standard 2-simplex

Let $v_0, v_1, \dots, v_n \in \mathbb{R}^N$, $N \ge n+1$

Define $[v_0, v_1, \dots, v_n] \equiv f : \Delta_n \to \mathbb{R}^N$

$$f\left(\sum_{i=0}^{p} \lambda_i e_i\right) = \sum_{i=0}^{p} \lambda_i v_i \qquad (\text{hence } f(e_i) = v_i)$$

$[v_0, v_1, \dots, v_n]$ is called affine singular simplex; its image is the convex envelope of the v_i's

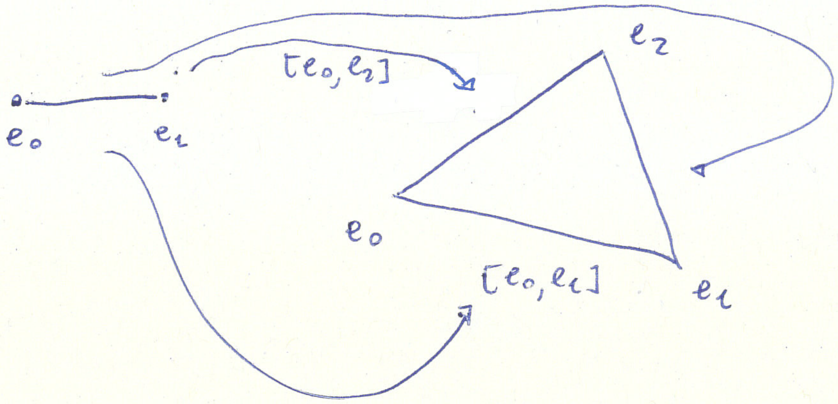


In particular

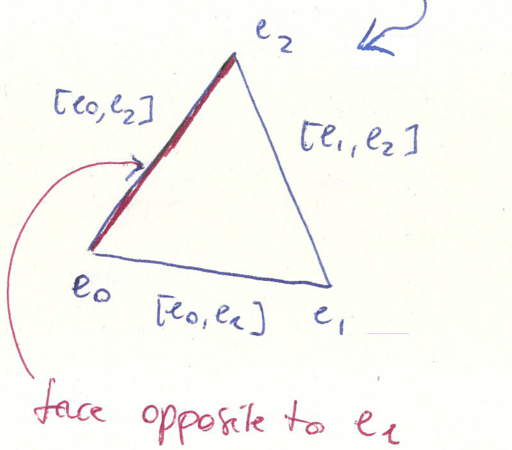
$$[e_0, e_1, \dots, \overset{\wedge}{e_i}, \dots e_p] : \Delta_{p-1} \longrightarrow \Delta_p$$

omitted the map

is called $\underline{i^{th}}$ - $\underline{\text{face map}}$, and it is more compactly denoted by F_i^p (it is the "face" opposite to the vertex e_i)



more compactly



Explicitly

$$F_i^p(e_k) = \begin{cases} e_k, & k < i \\ e_{k+1}, & k \geq i \end{cases}$$

example $\quad F_1^2 = [e_0, e_2]$: $\quad \begin{array}{l} e_0 \longmapsto e_0 \\ e_1 \longmapsto e_2 \end{array}$

Notice that

$$j > i \implies F_j^{p+1} \circ F_i^p = [e_0, \dots, \hat{e}_i, \dots \hat{e}_j, \dots e_p]$$

$$j \leq i \implies F_j^{p+1} \circ F_i^p = [e_0, \dots \hat{e}_j \dots \hat{e}_{i+1} \dots e_p]$$

(both are maps $\Delta_{p-1} \longrightarrow \Delta_{p+1}$

Therefore (and this is will be crucial)

$$j \leq i \implies \boxed{F_j^{p+1} \circ F_i^p = F_{i+1}^{p+1} \circ F_j^p = [e_0 .. \hat{e}_j .. \hat{e}_{i+1} .. e_p]}$$

$(j < i+1)$

* Singular homology.

Let X be a topological space; a continuous map

$$\sigma_p : \Delta_p \longrightarrow X$$

standard p-simplex

is called singular p-simplex.

The free abelian group generated by the singular p-simplices is called p^{th}-singular chain group (notation: C_p or $\Delta_p(X)$)

A singular p-chain is then a formal (finite) sum

$$c = \sum_{\sigma = \text{p-simplex}} n_\sigma \cdot \sigma \quad , \quad n_\sigma \in \mathbb{Z}$$

Let $\sigma : \Delta_p \to X$ a singular p-simplex. Set

$$\sigma^{(i)} = \sigma \circ F_i^p \equiv i^{th}\text{-face of } \sigma$$

The boundary of σ is (definition) the following $(p-1)$-chain

$$\partial_p \sigma := \sum_{i=0}^{p} (-1)^i \sigma^{(i)}$$

boundary of σ. Define, for a p-chain $c = \sum n_\sigma \sigma$, its boundary ∂c via:

$$\boxed{\partial_p c := \sum_\sigma n_\sigma \cdot \partial_p \sigma}$$

Amplification

More precisely, let $\Sigma = \{\sigma_\alpha\}_{\alpha \in \Omega}$ be any set

index set

define

$$f_\alpha : \Sigma \longrightarrow \mathbb{Z}$$

$$f_\alpha(\sigma_\beta) = \delta_{\alpha\beta} \quad \left(= \begin{cases} 1 & \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases} \right)$$

Let

$$\langle \sigma_\alpha \rangle_{\alpha \in \Omega} = \left\{ \sum_\alpha n_\alpha f_\alpha \;\middle|\; n_\alpha \neq 0 \text{ for a finite set of indices} \right\}$$

$$\sum n_\alpha f_\alpha : \Sigma \longrightarrow \mathbb{Z}$$

Functions of this type can be naturally added, and give rise to an abelian group, called the free abelian group generated by Σ (i.e. the σ_α)

abridged notation:

$$\sum_\alpha n_\alpha \sigma_\alpha$$

Hence

$$\partial_p : \Delta_p(X) \longrightarrow \Delta_{p-1}(X)$$

is actually a group homomorphism.

The crucial remark is the following:

$$\boxed{\partial_p \circ \partial_{p+1} = 0}$$

↖ the zero-chain ($n_\sigma \equiv 0$)

or, briefly $\boxed{\partial^2 = 0}$

<u>Proof</u>. It is enough to take $c = \sigma$ a $(p+1)$-simplex

Then:

$$(\partial_p \circ \partial_{p+1})\,\sigma = \partial_p(\partial_{p+1}\sigma) = \partial_p\left[\sum_{j=0}^{p+1}(-1)^j(\sigma\circ F_j^{p+1})\right]$$

$$= \sum_{j=0}^{p+1}(-1)^j \sum_{i=0}^{p}(-1)^i(\sigma\circ F_j^{p+1})\circ F_i^p =$$

$$= \sum_{j=0}^{p+1}\sum_{i=0}^{p}(-1)^{i+j}(\sigma\circ F_j^{p+1}\circ F_i^p) =$$

$$\underbrace{\sum_{0\le i<j\le p+1}(\quad)}_{A} + \underbrace{\sum_{0\le j\le i\le p}(\quad)}_{B}$$

We now check that $B = -A$. Indeed

$$B = \sum_{0\le j\le i\le p}(-1)^{i+j}\,\sigma\circ(F_j^{p+1}\circ F_i^p) = F_{i+1}^{p+1}\circ F_j^p$$

↖ recall

$$= \sum_{0\le j\le i\le p}(-1)^{i+j}\,\sigma\circ F_j^{p+1}\circ F_i^p$$

If we set $i' = i+1$ $\quad (i = i'-1)$

we have

$$B = \sum_{0 \le j < i' \le p+1} (-1)^{(i'+j-1)} \sigma \circ F^{p+1}_{i'} F^{p}_{j}$$

$= \; -A$ $\quad$ (via the following change of dummy indices : $i \mapsto j$, $j \mapsto i'$)

$\square$

If $p < 0$, set $\Delta_p(X) = 0$ $\quad$ (.. trivial group)

and $\partial_p = 0$ for $p \le 0$. Form the 'so-called

$(\Delta_*(X), \partial)$ <u>singular</u> <u>chain</u> <u>complex</u>

homomorphisms

$$\boxed{\cdots} \longrightarrow \Delta_{p+1}(X) \xrightarrow{\partial_{p+1}} \Delta_p(X) \xrightarrow{\partial_p} \Delta_{p-1}(X) \longrightarrow \boxed{\cdots}$$

abelian groups

c $\qquad$ 0 $\qquad$ $\boxed{\partial_p \partial_{p+1} = 0}$

in short

Set $\boxed{\operatorname{Ker} \partial_p := Z_p(X)}$ $\quad$ it is a <u>subgroup</u> of $\Delta_p(X)$, and its elements are called p-<u>cycles</u>

kernel $\quad \| \quad \{ c \in \Delta_p(X) / \partial_p c = 0 \}$

and $\boxed{\operatorname{Im} \partial_{p+1} := B_p(X)}$ $\quad$ This is also a subgroup of $\Delta_p(X)$, and <u>actually</u> of $Z_p(X)$, since $\partial^2 = 0$; its elements are called p-<u>boundaries</u>

$\star$ The quotient <u>group</u> ($B_p(X)$ is normal in $Z_p(X)$)

$$\boxed{H_p(X) := Z_p(X) / B_p(X) \equiv \operatorname{Ker} \partial_p / \operatorname{Im} \partial_{p+1}}$$

is called <u>p^{th}</u>- <u>singular</u> <u>homology</u> <u>group</u>

Their direct sum is called <u>singular homology</u> of X and denoted by $H_*(X) = H_*(\Delta_*(X))$ (homology of the complex $\Delta_*(X)$, see the chapter on homological algebra)

If chains are taken with coefficients in $\mathbb{R}$ (or $\mathbb{Q}$, or, more generally, in a field), then the $H_p(X)$ become vector spaces and their dimension

$$\boxed{h_p(X) := \dim H_p(X) =: b_p}$$

(better: $H_p(X, \mathbb{R})$)

is termed p^{th} - Betti number (in honour of Enrico Betti).

In general, b_p is the rank of the abelian group $H_p(X)$, i.e. the number of copies of $\mathbb{Z}$ in the canonical decomposition of a finitely generated abelian group G ($G = \underbrace{\mathbb{Z}^r}_{r\text{-copies of } \mathbb{Z}} \oplus$ torsion part)

r: rank

direct sum

finite group

Some additional details:

$$A = \underset{\substack{\text{free abelian group}}}{F} \oplus T \quad \text{torsion subgroup}$$

finitely generated abelian group

$F \cong \mathbb{Z}^r$, r = rank — free part — uniquely determined

$T \cong \underbrace{\bigoplus_{i \in \mathcal{Y}}}_{} \mathbb{Z}/(p_i)$ — uniquely determined

$\mathcal{Y}$: finite set; p_i: prime

may be reduced to the neutral element

$$A = \langle \sigma_1 \ldots \sigma_n \rangle = \{ \sigma_1^{m_1} \ldots \sigma_n^{m_n} \} \equiv \{ \sum m_i \sigma_i \}$$

generators

multiplicative notation

additive notation — better suited to the abelian case

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

A is abelian

Remark Singular homology is a generalization of simplicial homology, and it is more flexible of the latter. In general, efficient computation of homology groups requires sophisticated techniques.
In a subsequent lecture we shall treat simplicial homology in some detail, in order to get proper acquaintance with the basic techniques of algebraic topology and discuss many simple but fundamental examples