Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Prof. Mauro Spera

Dipartimento di Matematica e Fisica "Niccolò Tartaglia", UCSC, Brescia

Lecture XLII

FINITE DIMENSIONALITY OF DE RHAM COHOMOLOGY. DIGRESSION ON COMPLETELY INTEGRABLE SYSTEMS

1 Prelude to the de Rham theorem

Let us resort to the de Rham complex (M smooth manifold, $\dim M = n$)

$$\Lambda^{k-1}(M) \xrightarrow{d} \Lambda^{k}(M) \xrightarrow{d} \Lambda^{k+1}(M) \to \dots \qquad d^2 = 0$$

$$Z^k_{dR}(M) = \{ \omega \in \Lambda^k(M) \;/\; d\omega = 0 \} \quad \text{closed } k\text{-forms}$$

$$B^k_{dR}(M) = \{ \omega \in \Lambda^k(M) \;/\; \omega = d\alpha,\ \alpha \in \Lambda^{k-1}(M) \} \quad \text{exact } k\text{-forms}$$

$$\boxed{\begin{aligned} H^k_{dR}(M) &:= Z^k_{dR}(M) \big/ B^k_{dR}(M) \quad \rightsquigarrow \; k^{th} \text{ de Rham cohomology group} \\ h^k_{dR} &:= \dim H^k_{dR}(M) \end{aligned}}$$

We shall prove that if M admits a finite good covering (i.e. with contractible or empty intersections), then the spaces H^k_{dR} are indeed finite dimensional.

For simplicity, let us assume M compact, oriented.

for a smooth <u>chain</u> (defined in a natural way), one can define $\int_c \omega$ and Stokes' theorem still holds:

$$\int_{\partial c} \omega = \int_c d\omega$$

Now let c be a k-cycle ($\partial c = 0$) and ω a closed k-form ($d\omega = 0$).

✱ <u>Basic fact</u>:

$$\boxed{\int_{c+\partial b} (\omega + d\alpha) = \int_c \omega}$$

Indeed, this follows immediately from <u>Stokes' theorem</u>:

$$\int_{c+\partial b} (\omega + d\alpha) = \int_c \omega + \underbrace{\int_c d\alpha}_{①} + \underbrace{\int_{\partial b} \omega}_{②} + \underbrace{\int_{\partial b} d\alpha}_{③}$$

① : $\displaystyle\int_c d\alpha \overset{\text{Stokes}}{=} \int_{\underset{\overset{\shortparallel}{0}}{\partial c}} \alpha = 0$ ② : $\displaystyle\int_{\partial b} \omega = \int_b \overset{\overset{0}{\shortparallel}}{d\omega} = 0$

③ : $\displaystyle\int_{\partial b} d\alpha = \int_{\underset{\overset{\shortparallel}{0}}{\partial^2 b}} \alpha = 0$, or, equivalently

$\partial^2 = 0$

$$\int_{\partial b} d\alpha = \int_b d^2\alpha = 0$$

$d^2 = 0$

Therefore, $\int_c \omega$ <u>only depends on</u> $[\omega] \in H^k(M)$ and $[c] \in H_k(M)$ (that is, on <u>cohomology</u> and <u>homology</u> classes, respectively)

Now take chains with <u>real</u> coefficients (so homology groups become vector spaces)

One can define the following <u>pairing</u>

$$\left[\begin{array}{ccc} H_p(M) \times H^p_{dR}(M) & \longrightarrow & \mathbb{R} \\ ([c],[\omega]) & \longmapsto & \int_c \omega \end{array}\right]$$

(well-defined, in view of the previous remarks). Set

$$\left[\begin{array}{cccc} \psi_{[\omega]} : & H_p(M) & \longrightarrow & \mathbb{R} \\ & [c] & \longmapsto & \int_c \omega \end{array}\right]$$

obviously $\psi_{[\omega]} \in H_p(M)^*$ (dual of $H_p(M)$)

4.4 <u>Theorem</u> (de Rham) $H^p_{dR}(M) \cong H_p(M)$ (finite dimensional vector spaces)

(it may be proved in many different ways) A proof will be given in a forthcoming lecture

Let us confine ourselves, for the time being, to a few comments

One has to show that

$$\begin{array}{ccccc} \psi : & [\omega] & \longmapsto & \psi_{[\omega]} \\ & \cap\!\!\!\!\mid & & \cap \\ & H^p_{dR}(M) & & H_p(M)^* \end{array}$$

is <u>bijective</u>.

• Injectivity $\psi_{[\omega]} = 0 \Rightarrow [\omega] = 0$

That is $\int_C \omega = 0 \quad \forall$ p-cycle $C \Rightarrow \omega$ is exact

• surjectivity : any element in $H_p(M)^*$ is of the form $\psi_{[\omega]}$ for some $[\omega]$.

Let $(C_i)_{i=1,2,\dots b_p}$ be a basis of p-cycles in $H_p(M)$.

(b_p = p^th^ Betti number). Let $(\alpha_1, \alpha_2, \dots \alpha_{b_p}) \in \mathbb{R}^{b_p}$.

Then $\exists \; [\omega] \in H^p_{dR}(M)$ such that $\int_{C_i} \omega = \alpha_i$

$i = 1, 2, \dots, b_p$

✱ The numbers $\int_{C_i} \omega$ are called periods of the closed p-form ω (actually, of its class).

Therefore, surjectivity means that given the α_i's, they are the periods of a closed p-form ω.

So, synthetically, the de Rham theorem can be stated as follows:

given $\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, b_p$,
$\exists ! \; [\omega] \in H^p_{dR}(M)$ having the α_i's as periods

That is, periods may be prescribed at will, and the cohomology class is uniquely determined.

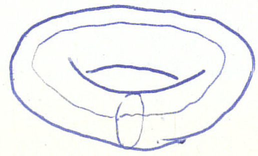
4 A brief mechanical digression

$\dim M = 2n$

Let (M, ω, H) be a completely integrable Hamiltonian system, let $[\omega] = 0$ i.e. $\omega = d\theta$ globally (θ symplectic potential   example $\omega = dp \wedge dq$ $\theta = p\,dq$)

Let Λ be a Liouville torus ($\dim \Lambda = n$) $\Lambda \simeq \underbrace{S^1 \times \ldots S^1}_{n \text{ copies}}$

$H_1(\Lambda, \mathbb{R}) \cong \mathbb{R}^n$



(c_i), $i = 1 \ldots n$ basis of 1-cycles.

Then $\omega\big|_\Lambda \equiv 0 \Rightarrow d\theta\big|_\Lambda = 0 \Rightarrow d(\theta\big|_\Lambda) = 0$

$\Rightarrow \quad \theta \equiv \theta\big|_\Lambda$ is a closed 1-form.

$\Rightarrow [\theta] \in H^1_{dR}(\Lambda)$. The periods of θ,

$$\boxed{I_i := \frac{1}{2\pi}\int_{c_i} \theta}$$

yield precisely the action variables (which label Liouville tori)

So $\theta \sim \sum I_k \, d\varphi_k$ cohomologous   angular variables

$\int_{c_j} d\varphi_i = 2\pi \delta_{ij}$ i^{th} angular form

So one may choose $\theta = \sum I_i \, d\varphi_i$