– Lectures on

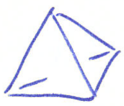
DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

Prof. Mauro SPERA – Dipartimento di Matematica e Fisica "Niccolò Tartaglia", UCSC, Brescia

Lecture XLIV — COMPUTATION OF HOMOLOGY GROUPS

★ Computation of some homology groups $\quad ((H_{sing})_* = (H_{simpl})_*)$

1. The sphere S^2


$\approx$


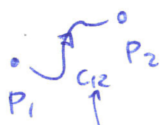
$$\begin{cases} H_0(S^2) \cong \mathbb{Z} \\ H_1(S^2) \cong 0 \\ H_2(S^2) \cong \mathbb{Z} \end{cases}$$

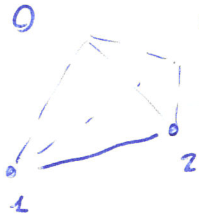
Poincaré duality

$\boxed{H_0}$ 0-chains: $\quad c = \sum_i n_i \cdot p_i$, $n_i \in \mathbb{Z}$ ← points

$\partial p_i = 0 \quad \Rightarrow \quad \partial c = 0 \quad$ (c : 0-cycles)

$p_1 \sim p_2$





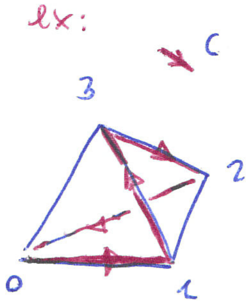
$p_2 - p_1 = \partial c_{12}$

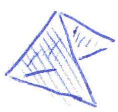
$\Rightarrow \quad p \sim p_0$ ← fixed $\quad \Rightarrow \quad c \sim n\, p_0$, $n \in \mathbb{Z}$

$\Rightarrow \quad H_0(S^2) \cong \mathbb{Z} \quad$ ($\mathbb{R}$, if reals are used as coefficients)

$\boxed{H_1}$ Every cycle is a boundary of a 2-chain

ex:





Algebraically:

$$C = \langle 0,1\rangle + \langle 1,3\rangle + \langle 3,2\rangle + \langle 2,0\rangle$$

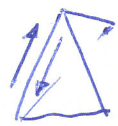
$$(\partial C = \langle 1\rangle - \langle 0\rangle + \dots + \langle 0\rangle - \langle 2\rangle = 0)$$

Claim: $C = \partial C'$, $C' = \langle 0,1,3\rangle - \langle 0,2,3\rangle$

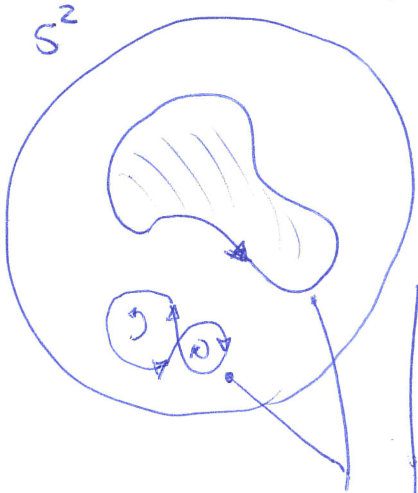
In fact: $\partial C' = \partial(\langle 0,1,3\rangle - \langle 0,2,3\rangle) =$

$$= \langle 1,3\rangle - \cancel{\langle 0,3\rangle} + \langle 0,1\rangle$$
$$- \langle 2,3\rangle + \cancel{\langle 0,3\rangle} - \langle 0,2\rangle$$

cancellation



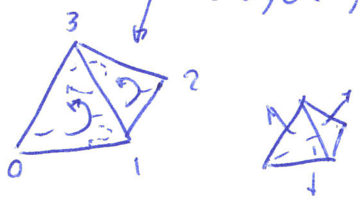
$$= \langle 1,3\rangle + \langle 0,1\rangle + \langle 3,2\rangle + \langle 2,0\rangle = C$$



singular homology viewpoint

$H_1(S^2) \cong 0$

$\boxed{H_2}$



a 2 cycle, which is not a boundary

$$C = \langle 0,1,3\rangle + \langle 1,2,3\rangle + \langle 0,3,2\rangle + \langle 0,2,1\rangle$$

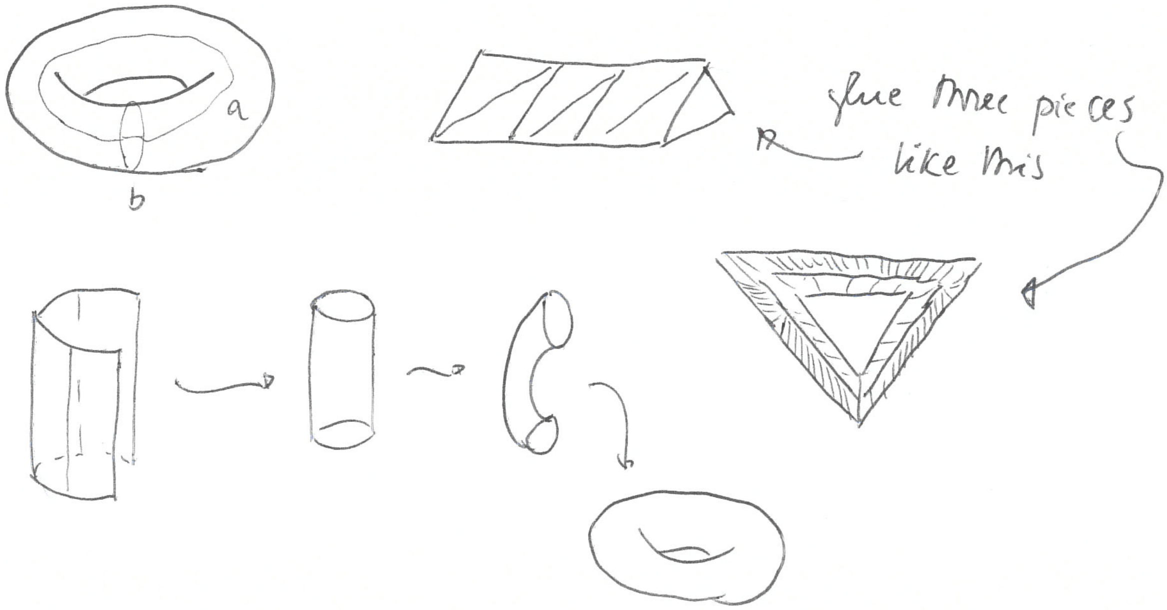
$$\begin{aligned}\partial C = {} & \langle 1,3\rangle - \langle 0,3\rangle + \langle 0,1\rangle \\ & + \langle 2,3\rangle - \langle 1,3\rangle + \langle 1,2\rangle \\ & + \langle 3,2\rangle - \langle 0,2\rangle + \langle 0,3\rangle \\ & + \langle 2,1\rangle - \langle 0,1\rangle + \langle 0,2\rangle = 0\end{aligned}$$

basic idea: build up a 2-chain trying to insure cancellation of the adjacent 1-simplices

$$\Rightarrow H_2(S^2) \cong \mathbb{Z}$$

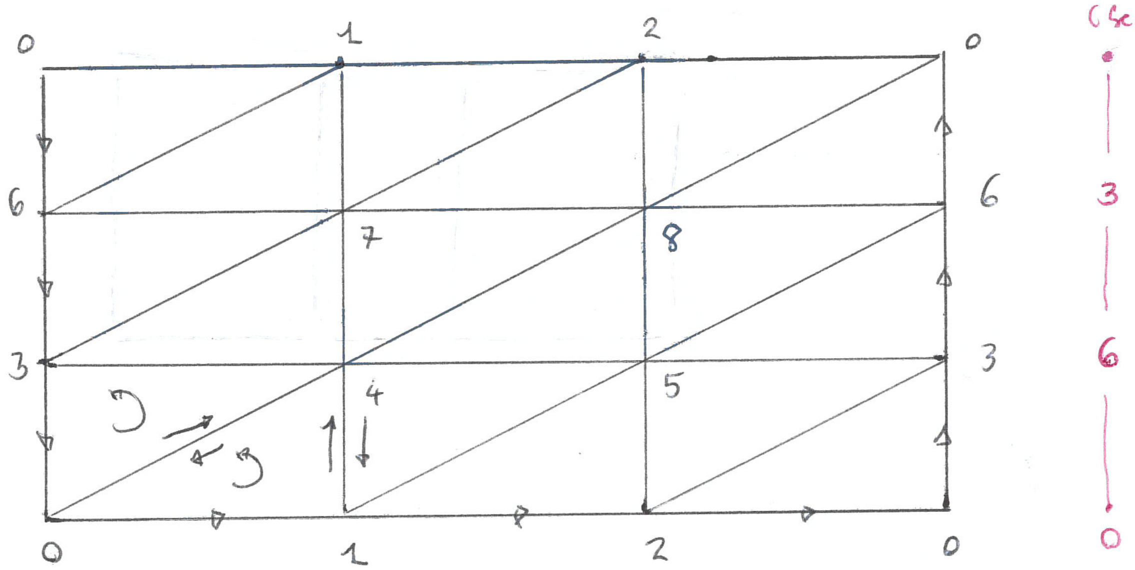
(every other 2-cycle is a multiple of C)

2. The torus $T^2 = S^1 \times S^1$



$$\begin{cases} H_0(T^2) \cong \mathbb{Z} & \text{... clear} \\ H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} \\ H_2(T^2) \cong \mathbb{Z} \end{cases}$$

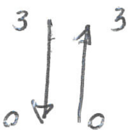
★ torus ★ Klein bottle (see below)



$\boxed{H_2}$ all 2-cycles are integral multiples of

$c = \sum$  (see picture)

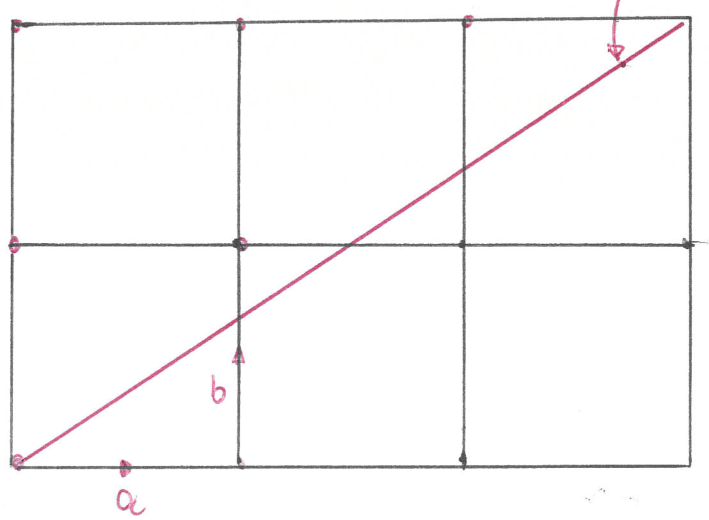
notice the good matching of the "external sides" as well, in addition to cancellation of the "internal" ones



One has no boundaries $\Rightarrow H_2(T^2) \cong \mathbb{Z}$

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ ~ lattice

$c = 3a + 2b$



wrapping c back on the torus one obtains a trefoil knot (in general, a toral knot)

$\gamma_{(p,q)}$ $\quad (p,q) = 1$ ← rel. prime

$H_1(\mathbb{T}^2) = \mathbb{Z}^2$ = free abelian group generated by a and b (actually, their homology classes)

3. <u>The</u> <u>Klein</u> <u>bottle</u> K $\qquad H_0(K) = \mathbb{Z}$ (clear)

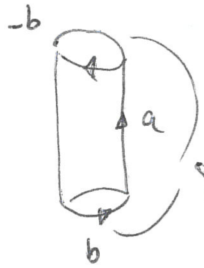
$H_2(K) = 0$. Indeed, if one tries to construct a 2-cycle as before, cancellation of the "external" sides fails:

→ This signals non orientability (for compact spaces, with no boundary)

$\overset{3}{\uparrow}\; \overset{3}{\uparrow}$ etc ...
$0 \quad 0$

$H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$

free part $\qquad$ torsion part



Indeed, upon glueing a stays the same, whereas b is to be identified with $-b$ $\Rightarrow$ $2b = 0$

$\Rightarrow H_1(K) = \langle a, b \,;\, b^2 = e, ab = ba \rangle$

generators $\qquad$ & relations $\qquad$ abelianness

$\cong \mathbb{Z} \oplus \mathbb{Z}_2$

multiplicative notation

$= \{ m\,a + [n]\,b \}$ $\qquad [n] \in \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$

additive notation

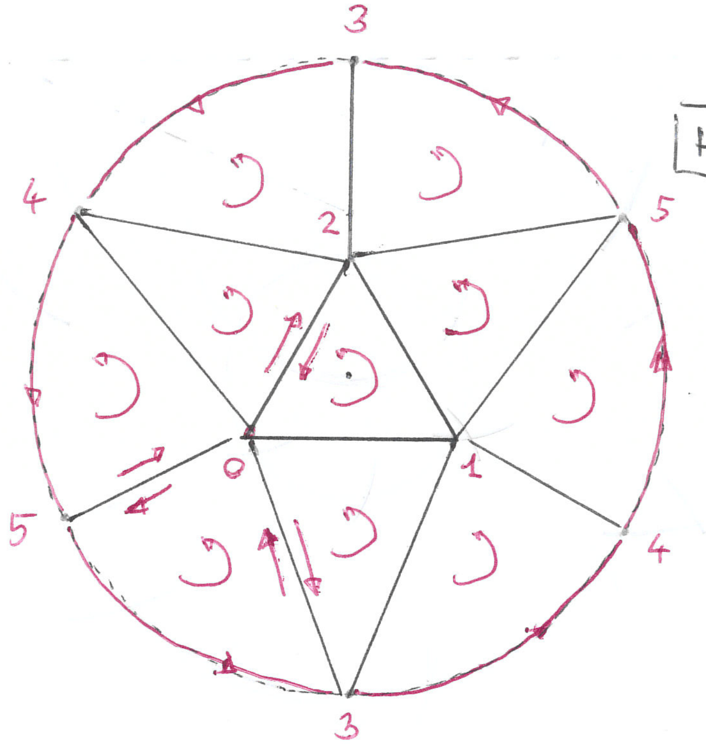
4. The real projective plane $\mathbb{RP}^2 \cong S^2/\sim$ ← antipodal pt identification

$\cong$ Closed disc with antipodal boundary pts identified

$H_0 \cong \mathbb{Z}$ clear

$H_1 \cong \mathbb{Z}_2$ ← torsion

$H_2 \cong 0$



$\boxed{H_2}$ try to construct a 2-cycle w

One has to ensure cancellation ↑↓ of the "internal" sides, first. So take the 2-chain

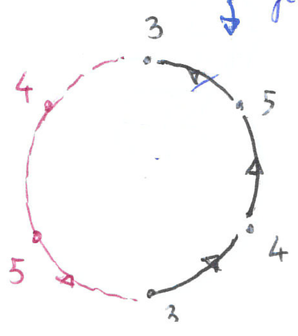
$w = n \sum \triangle$
$n \in \mathbb{Z}$

One has $\partial w = 2n \{ \langle 3,4\rangle + \langle 4,5\rangle + \langle 5,3\rangle \}$

$\Rightarrow \partial w = 0 \Rightarrow n = 0$. Therefore $\boxed{H_2 \cong 0}$

$\boxed{H_1}$ A brief inspection shows that any 1-cycle c is homologous to $n \{ \langle 3,4\rangle + \langle 4,5\rangle + \langle 5,3\rangle \}$ (for some $n \in \mathbb{Z}$), and that if $n = 2k$, then c is a boundary. Hence $\boxed{H_1 \cong \mathbb{Z}_2}$

also geometrically clear



★ <u>Comment</u> : The <u>Hurewicz</u> <u>theorem</u> tells us that, in general

$$H_1(X) \cong \frac{\pi_1(X,*)}{[\pi_1(X,*),\pi_1(X,*)]}$$

base point

commutator subgroup

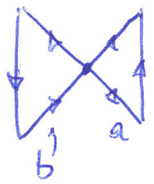
that is : $\boxed{H_1 \text{ is } \pi_1 \text{ made abelian}}$

$[G,G]$ = commutator subgroup = group generated by commutators $ghg^{-1}h^{-1}$

(it is a normal subgroup of G $\Rightarrow$ $G/[G,G]$ is a group as well)

In our examples, π_1 was already abelian. <u>This is not always so</u>:

" bouquet "

X = 

$$\pi_1(X) = \underset{\pi_1(\triangleleft)}{\mathbb{Z}} * \mathbb{Z}$$

↖ <u>free</u> product

= free group gen by

↑ "no relations outside the bare group law"

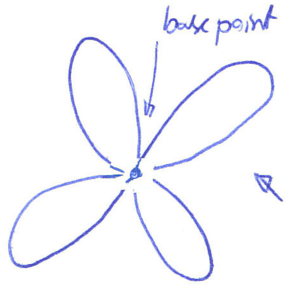
Abelianization of $\pi_1(X)$ gives $\mathbb{Z}^2 = \mathbb{Z}\oplus\mathbb{Z} \cong H_1(X)$

generators a, b

$ab \neq ba$

$ab^2a \neq a^2b^2$ etc...

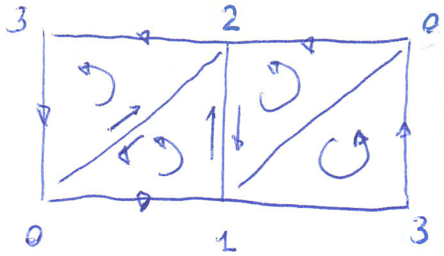
similarly

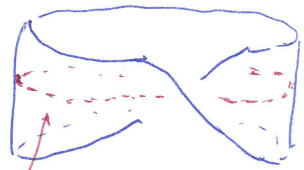
X = 

↖ n circles

$$\pi_1(X) = \underbrace{\mathbb{Z} * \mathbb{Z} * \dots \mathbb{Z}}_{n \text{ copies}}$$

5. The Möbius strip M

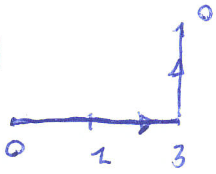


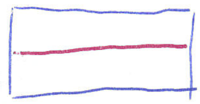


homotopically equivalent to a circle S^1

$$\left\{\begin{array}{l} H_0(M) \cong \mathbb{Z} \\ H_1(M) \cong \mathbb{Z} \\ H_2(M) \cong 0 \end{array}\right.$$

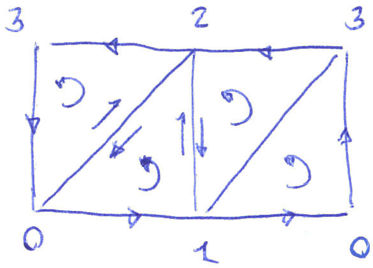
$\rightsquigarrow$ generator





In trying to build up a 2-cycle, the horizontal "sides" cannot be matched by anything (and also notice that $\overset{3}{\downarrow}_{0}\ \overset{0}{\uparrow}_{3}$ yields another conflict)

6. The cylinder



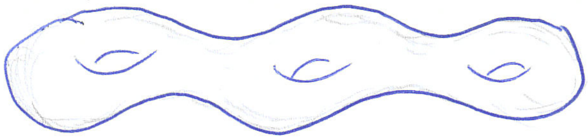
* Same result

(although $\overset{3}{\downarrow}_{0}\ \overset{3}{\uparrow}_{0}$ now yields no problem)

Aside

7. Sphere with g handles Σ_g

(connected sum of g tori)



$g = 3$

$g = 2$

generators ... relators

$$\pi_1(\Sigma_g) = \langle a_1 b_1, a_2 b_2 \ldots a_g b_g \;/\; a_1 b_1 a_1^{-1} b_1^{-1} \cdot a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle$$

$$\prod_{i=1}^{g} [a_i b_i] = 1$$

$H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$

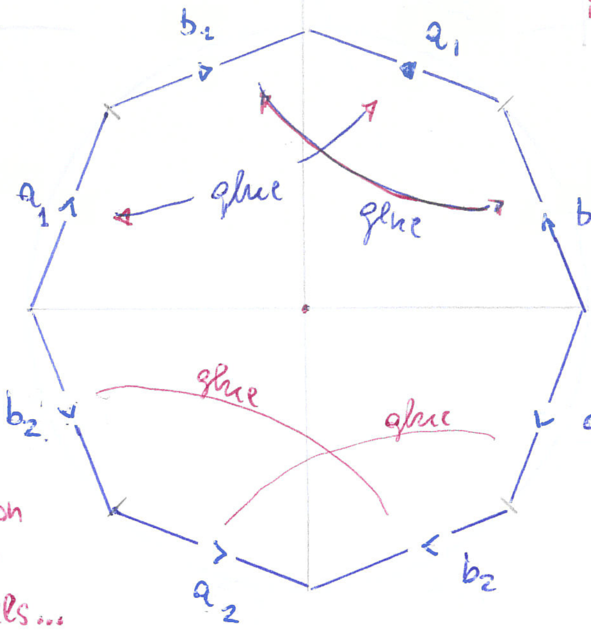
(abelianization)

$H_0(\Sigma_g) \cong \mathbb{Z}$

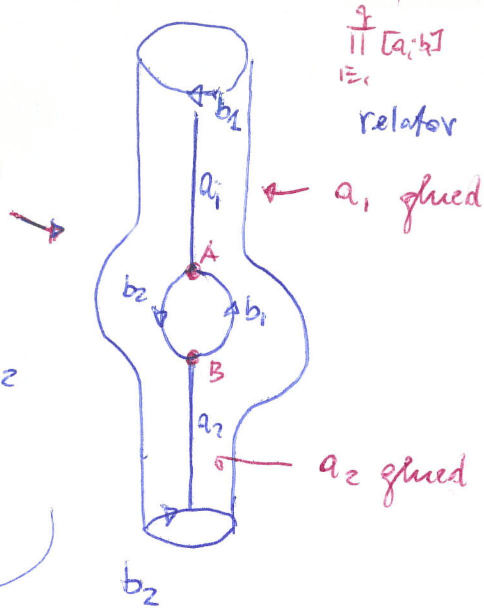
$H_2(\Sigma_g) \cong \mathbb{Z}$

also directly from a triangulation

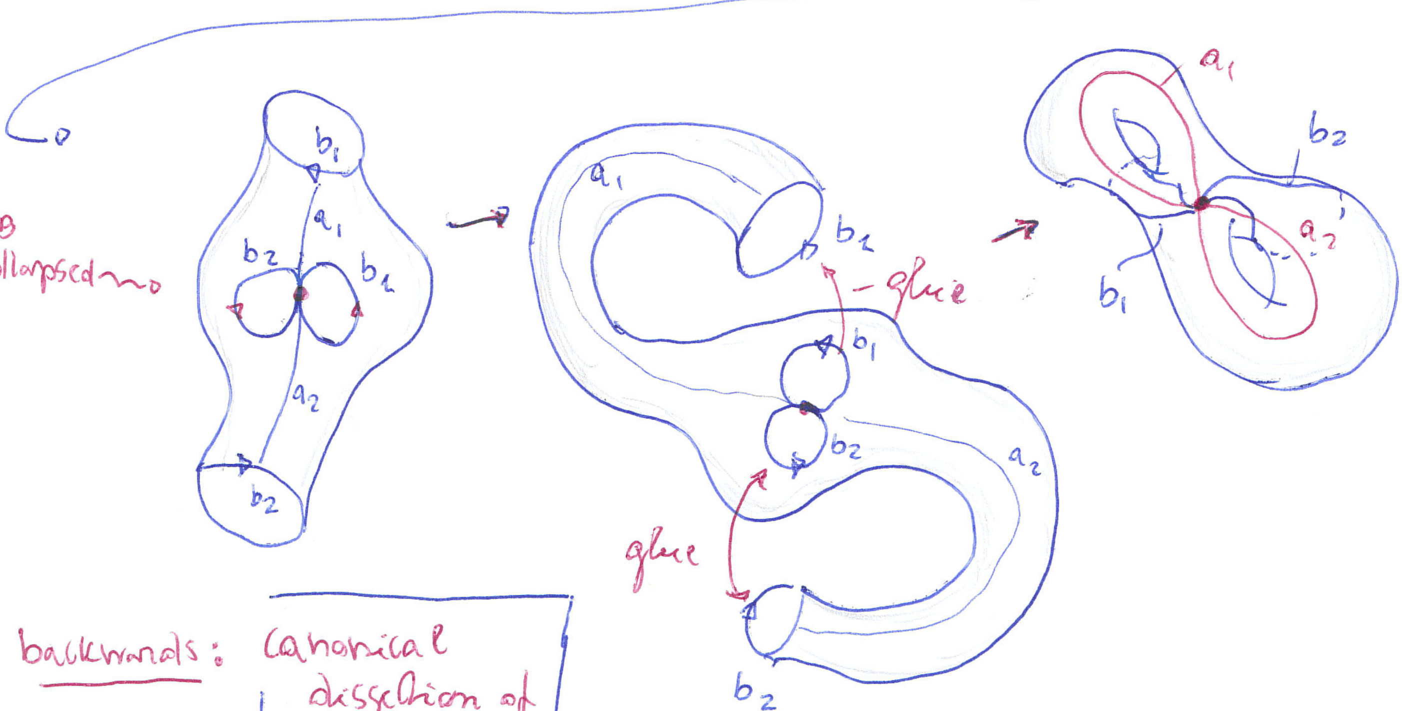
work out the details...



one writes also $\prod_{i=1}^{g} [a_i b_i]$ relator



A, B collapsed



backwards: canonical dissection of Σ_g