# Lecture XLIX

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## MAYER-VIETORIS CALCULATIONS (continued)

★ Further MV - calculations

$\Sigma_g$ : connected closed (= compact + no boundary) orientable surface (connected sum of $g$ tori)

$g = 0$ : sphere

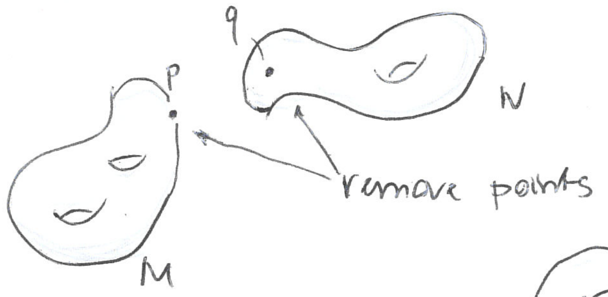
Let us prove that $\boxed{\chi(\Sigma_g) = 2 - 2g}$

(we have already shown this for $g = 0, 1$, and we hinted at the general case via the canonical dissection).

Given that $H^0(\Sigma_g) \cong H^2(\Sigma_g) \cong \mathbb{R}$ (e.g. by Poincaré duality for compact oriented manifolds), it would easily follow that

$\boxed{H^1(\Sigma_g) \cong \mathbb{R}^{2g}}$

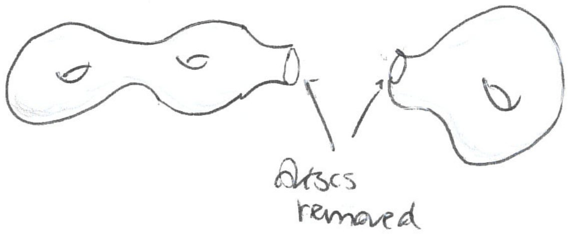
Given manifolds $M$, $N$, recall how to define their connected sum:



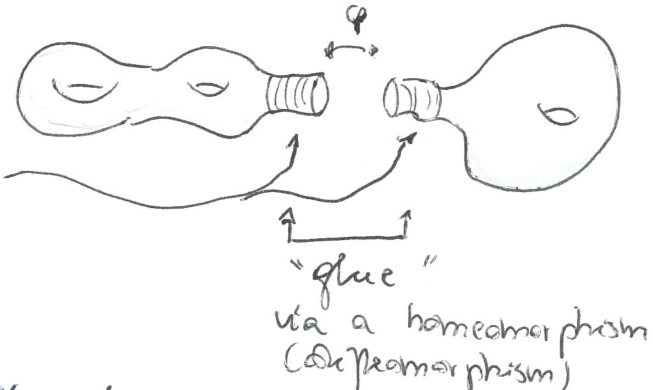

$U := M \setminus \{p\} \simeq M \setminus B_n$

$V := M \setminus \{q\} \simeq M \setminus B_n$

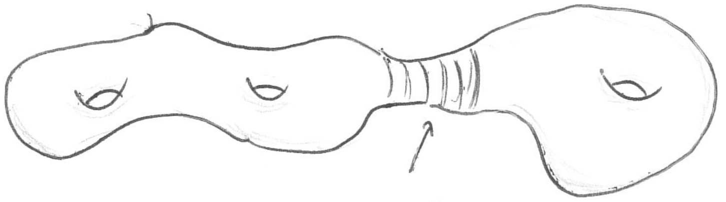
open $n$-dim ball




glue along a collar $\approx S^{n-1} \times \mathbb{R}$



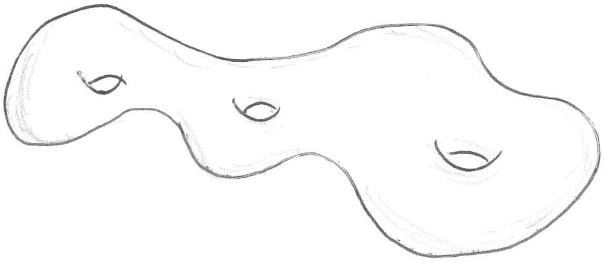
$U \cup V$



$U \cap V$

$\approx S^{n-1} \times \mathbb{R}$ "cylinder

$\approx$



( up to homeomorphisms (actually diffeomorphisms), the procedure is independent of all choices.. )

In our case ( dim = 2 ) we have

$U = M \setminus \{p\}, \quad V = N \setminus \{q\} \qquad U \cup V \approx M \# N$

$U \cap V \approx S^1 \times \mathbb{R}$ cylinder

After these preparations, consider the MV - sequence ( long exact )

$$0 \to H^0(M \# N) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \xrightarrow{d^*}$$

$$\xrightarrow{d^*} H^1(M \# N) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \xrightarrow{d^*}$$

$$\xrightarrow{d^*} H^2(M \# N) \to H^2(U) \oplus H^2(V) \to H^2(U \cap V) \to 0$$

Now $\boxed{h^0(U \cap V) = h^1(U \cap V) = 1 \ , \ h^2(U \cap V) = 0}$

Also recall, that for an exact sequence of vector spaces $\sum (-1)^l a_l = 0$ ($a_l$: dim)

Applying this remark to our sequence we immediately have:

notice this

$$\overbrace{h^0(U\cap V)}^{1}\ \overbrace{-\,h^1(U\cap V)}^{-1}\ \overbrace{+\,h^2(U\cap V)}^{0}$$

$$\chi(M \# N) - \chi(\overline{U}) - \chi(\overline{V}) + 0 = 0\ , \text{ i.e.}$$

$$\boxed{\chi(M \# N) = \chi(\overline{U}) + \chi(\overline{V})}$$

Now let $N = S^2$. Then $\overline{V} = S^2 \setminus \{q\} \approx \mathbb{R}^2$

$$\Rightarrow \chi(\overline{V}) = 1$$

Also $M \# S^2 \approx M \Rightarrow$

$$\chi(M) = \chi(\overline{U}) + 1$$

Similarly (for generic $N$): $\chi(N) = \chi(\overline{V}) + 1$

$$\Rightarrow \boxed{\chi(\overline{U}) = \chi(M) - 1}\ ,\ \boxed{\chi(\overline{V}) = \chi(N) - 1}.$$

Therefore

$$\boxed{\chi(M \# N) = \chi(M) + \chi(N) - 2} \quad (\bigstar)$$

We now argue by induction: since $\Sigma_g = \underbrace{\Sigma_1 \# \ldots \# \Sigma_1}_{g \text{ copies}}$ $(g \geq 1)$

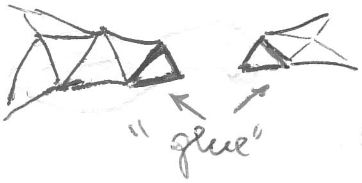
given the result for $g=1$ (already proven)

we have $\chi(\Sigma_g) = \chi(\Sigma_{g-1} \# \Sigma_1) = 2 - 2(g-1) - 2$

$$= 2 - 2g \qquad \boxed{\chi(\Sigma_g) = 2 - 2g}$$

$\square$

$\bigstar$ Remark. Notice that formula $(\bigstar)$ is immediate in terms of triangulations: in performing the connected sum (see fig.) we "lose" 2 faces, 3 vertices and 3 edges, but the last two objects enter the Euler-Poincaré characteristic with opposite signs, and the assertion follows.



★ Let $M$ be a compact, connected orientable manifold, with no boundary.

Then (e.g. via Poincaré Duality and $H^0(M) \cong \mathbb{R}$)

$$\boxed{H^n(M) \cong \mathbb{R}}$$

As a generator of the cohomology we may take a volume form $\omega$ (i.e. a never vanishing $n$-form, giving the orientation: choose it in such a way that $\int_M \omega > 0$ (the integral is finite)

[Clearly, we may set the integral to 1, after multiplying by a suitable constant]

In fact, we show that $[\omega] \neq 0$.

By contradiction, if $[\omega]$ were equal to 0, then $\omega = d\alpha$, for $\alpha \in \Lambda^{n-1}(M)$. Moreover, by Stokes' theorem:

$$0 < \int_M \omega = \int_M d\alpha \underset{\text{Stokes}}{=} \int_{\underset{\overset{\|}{0}}{\partial M}} \alpha = 0,$$

* as a chain...

yielding a contradiction.

★ Theorem : There are no symplectic structures on $S^4$

Recall that a symplectic manifold $(M, \omega)$ is a smooth manifold equipped with a closed, non degenerate 2-form $\omega$ ( we already know then that $\dim M =$ even)

Proof. Let $\omega$ be a symplectic form on $S^4$ : $d\omega = 0$

Then $[\omega] \in H^2(S^4) = \{0\}$ ( recall...),

i.e. $\omega = d\alpha$ , $\alpha \in \Lambda^1(S^4)$

Now

$$\omega \wedge \omega = d\alpha \wedge \omega = d(\alpha \wedge \omega) + \alpha \wedge \underset{0}{\underset{\parallel}{d\omega}}$$

$$= d(\alpha \wedge \omega)$$

But $\omega \wedge \omega$, in view of non degeneracy of $\omega$, never vanishes, so it is a volume form, hence it cannot be exact, by the preceding observation.

$$\left( 0 \neq \int_{S^4} \omega \wedge \omega = \int_{S^4} d(\alpha \wedge \omega) = \int_{\underset{=0}{\partial S^4}} \alpha \wedge \omega = 0 \right)$$

★ Let us compute the cohomology of

$$\boxed{\mathbb{R}^3 \setminus \{(0,0,0)\}}$$

Clearly $\mathbb{R}^3 \setminus \{(0,0,0)\} \underset{\text{Diffeo}}{\simeq} S^2 \times \mathbb{R}$

$$\Rightarrow \quad \boxed{H^*(\mathbb{R}^3 \setminus \{(0,0,0)\}) \cong H^*(S^2) = \begin{cases} \mathbb{R} & q=0 \\ 0 & q=1 \\ \mathbb{R} & q=2 \\ 0 & q=3 \end{cases}}$$

notice this

additional remarks:

$H^0(\mathbb{R}^3 \setminus \{0\}) \cong \mathbb{R}$ connectedness

$H^3(\mathbb{R}^3 \setminus \{0\}) \cong 0$ non compactness

$H^1(\mathbb{R}^3 \setminus \{0\}) \cong 0$ is straightforward, since $\mathbb{R}^3 \setminus \{0\}$ is simply connected, and one can use Poincaré lemma.

As a generator for the 2nd cohomology group one can take

$$\boxed{E = \frac{1}{4\pi} \cdot \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}}$$

$= \boxed{\underline{E} \cdot \underline{n}\, d\sigma}$ = " flux element of an electric field generated by a point charge placed at the origin (charge value $= \frac{1}{4\pi}$ (normalization) "


$$\boxed{\underline{E} = \frac{1}{4\pi} \frac{\underline{r}}{\|\underline{r}\|^3}}$$

★★ This is the content of Gauss' theorem in electrostatics


$\iint_S \underline{E} \cdot \underline{n}\, d\sigma = 4\pi \cdot q$ (charge)

$\iint_{S'} \underline{E} \cdot \underline{n}\, d\sigma = 0$