Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

Prof. Mauro Spera – Dipartimento di Matematica e Fisica "Niccolò Tartaglia" - UCSC, Brescia

Lecture XLV

TOOLS IN HOMOLOGICAL ALGEBRA

◇ Elements of homological algebra

* Differential complexes

Let
$$C = \bigoplus_{q \in \mathbb{Z}} C^q$$
vector spaces

The C^q are equipped with maps:

$$\cdots \longrightarrow C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \longrightarrow \cdots$$

such that $d^2 = 0$

abuse of notation

The pair (C, d) is termed differential complex, and d is the differential of the complex (collectively)

One forms the cohomology of the complex (C, d):

cohomology groups

$$\boxed{H(C) = \bigoplus_{q \in \mathbb{Z}} H^q(C) \qquad H^q(C) := \frac{\operatorname{Ker} d \cap C^q}{\operatorname{Im} d \cap C^q}}$$

Comment

One abstracts common features encountered in dealing with de Rham and singular theories and builds up a general, far reaching machinery. Recall that the aim of algebraic topology is to find algebraic invariants (groups, algebras etc.) aimed at capturing topological properties and able to distinguish non homeomorphic spaces. Schematically

topological space "functor" algebraic invariant

$$X \longrightarrow I(X)$$

$$X \underset{\text{homeomorphic}}{\approx} Y \Rightarrow I(X) = I(Y)$$

$$(\Leftrightarrow I(X) \neq I(Y) \Rightarrow X \not\approx Y)$$

In general, it may well happen that $I(X) = I(Y)$ but $X \not\approx Y$.

* Chain maps

Given differential complexes (A, d_A), (B, d_B),

a homomorphism $f : A \to B$

is called a <u>chain map</u> if $\boxed{f \circ d_A = d_B \circ f}$

i.e., if, at each stage, one has commutative diagrams

$$\boxed{\begin{tikzcd} \arrow[r] & A^q \arrow[r, "d_A"] \arrow[d, "f"] \arrow[dr, bend right=60, dashed, "d_B \circ f"'] & A^{q+1} \arrow[r] \arrow[d, "f"] & {} \\ \arrow[r] & B^q \arrow[r, "d_B"'] & B^{q+1} \arrow[r] & {} \end{tikzcd}}$$

$f \circ d_A$

$d_B \circ f$ they agree

* Exact sequence of vector spaces and homomorphisms

$$\boxed{\longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow}$$

Given a sequence of vector spaces and homomorphisms, it is called <u>exact</u> if, $\forall i$,

$$\boxed{\operatorname{Ker} f_i = \operatorname{Im} f_{i-1}}$$

(the kernel of a map coincides with the image of the preceding one)

||| Cohomology groups measure the obstruction against exactness of a differential complex (at each stage)

Lemma. Consider an exact sequence of finite dimensional vector spaces

$$0 \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots A_n \xrightarrow{f_n} 0$$

Then, setting $a_i := \dim A_i$, one has

$$\boxed{\sum_{i=0}^{n} (-1)^i a_i = 0}$$

Proof. Just a simple application of the "nullity + rank" theorem ($T : V \to W$ homom. of vector spaces $\dim V < \infty$

$\dim \ker T + \dim \operatorname{Im} T = \dim V$)

Indeed, let
$$a_\ell = \underset{\dim \ker f_\ell}{\overset{\|}{a_\ell^{\ker}}} + \underset{\dim \operatorname{Im} f_\ell}{\overset{\|}{a_\ell^{\operatorname{Im}}}}$$

one has

$$\sum_{\ell=0}^{n} (-1)^\ell a_\ell = \sum_{\ell=0}^{n} \left\{ (-1)^\ell a_\ell^{\ker} + (-1)^\ell a_\ell^{\operatorname{Im}} \right\}$$

By exactness $a_{\ell-1}^{\operatorname{Im}} = a_\ell^{\ker}$, therefore via telescoping:

$$\underset{0}{\overset{\|}{a_0^{\ker}}} - a_1^{\ker} + a_2^{\ker} - \cdots$$

$$/\!/_{N+R} \qquad /\!/_{N+R}$$

$$+ a_0^{\operatorname{Im}} - a_1^{\operatorname{Im}} \qquad \pm \underset{0}{\overset{\|}{a_n^{\operatorname{Im}}}} \qquad = 0$$

More generally, given a complex A

$$0 \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} 0$$

$$f_i \circ f_{i-1} = 0 \qquad i = 0, \dots n$$

one forms $\quad H_i(A) = \ker f_i \big/ \operatorname{Im} f_{i-1} \quad$ ith- (co)-homology group

$h_i := \dim H_i$

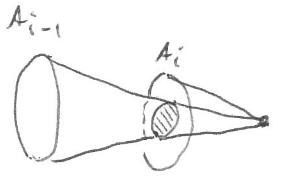
(measures obstruction to exactness at A_i)

One has the important

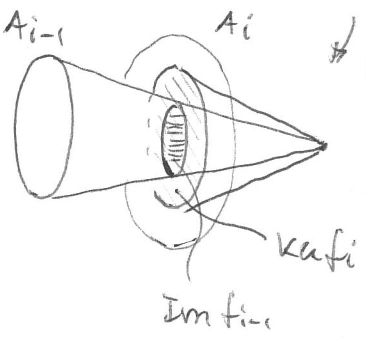
☆ Proposition

$$\sum_{l=0}^{n} (-1)^l a_l = \underbrace{\sum_{l=0}^{n} (-1)^l h_l}_{\equiv \chi(A)}$$

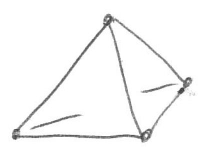
Euler-Poincaré Characteristic



exactness



Example

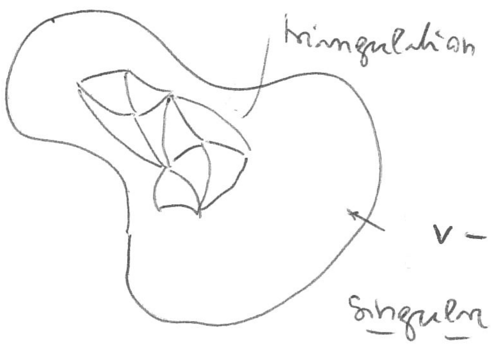


in simplicial homology:

$$V - E + F = 2$$

vertices, edges, faces

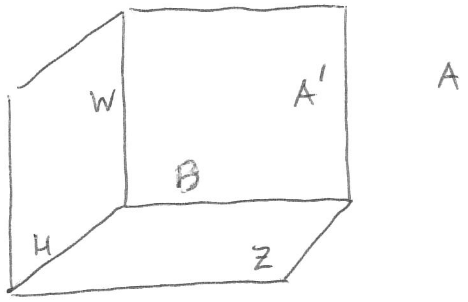
$$4 - 6 + 4 \qquad\qquad 2 = h_0 - h_1 + h_2, \quad h_0 = 1,\ h_1 = 0,\ h_2 = 1$$



$V - E + F = 2$

singular homology

Proof



$H \simeq Z/B$ identified $\leftrightarrow$ direct complement of B in Z

$Z = B \oplus H$

$$A_k = Z_k \oplus \overline{W}_k = H_k \oplus \underbrace{B_k \oplus \overline{W}_k}_{\cong\, A'_k}$$

$\overline{W}_k$: direct complement of Z_k in A_k

Then $a_k = a'_k + h_k$

$a_k = \dim A_k$, $a'_k = \dim A'_k$, $h_k = \dim H_k$

Now observe that

$$f_k(A'_k) = B_{k+1} \leq A'_{k+1}$$

$$f_k(\underset{Z_k}{\underset{\cap}{H_k}}) = \{0\}$$

setting $f'_k := f_k\big|_{A'_k}$ we get an exact subcomplex:

$$\boxed{\operatorname{Im} f'_k = B_{k+1} = \ker f'_{k+1}}$$

Thus

$$\chi(A) = \sum_{k=0}^{n} (-1)^k a_k = \sum_{k=0}^{n} (-1)^k (a'_k + h_k)$$

$$= \underbrace{\sum_{k=0}^{n} (-1)^k a'_k}_{\substack{= 0 \\ \text{preceding lemma}}} + \sum_{k=0}^{n} (-1)^k h_k$$

and we are done. $\square$

* short exact sequence of differential complexes A, B, C
(and chain maps f, g)

$$\boxed{0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0}$$

This means : $\begin{cases} f \text{ injective} \\ g \text{ surjective} \\ \operatorname{Im} f = \operatorname{Ker} g \end{cases}$

** Theorem Given $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ as above,
there exists a long exact sequence in cohomology :

$$\begin{tikzcd}
\cdots \arrow[r] & H^{q}(A) \arrow[r, "f^*"] & H^{q}(B) \arrow[r, "g^*"] & H^{q}(C) \arrow[dll, "d^*"'] \\
 & H^{q+1}(A) \arrow[r, "f^*"] & H^{q+1}(B) \arrow[r, "g^*"] & H^{q+1}(C) \arrow[r, "d^*"] & \cdots
\end{tikzcd}$$

connecting homomorphism, (co)boundary homomorphism, Bockstein...

naturally induced by f and g

$\rightarrow$ Clearly, the crucial point is the definition of d^*, which is given by
* diagram chasing.

Proof. ① First define d^*

$$\begin{tikzcd}
 & da=0 & db & & \\
0 \arrow[r] & A^{q+1} \arrow[u, "d"] \arrow[r, "f"] & B^{q+1} \arrow[u, "d"] \arrow[r, "g"] & C^{q+1} \arrow[u, "d"] \arrow[r] & 0 \\
0 \arrow[r] & A^{q} \arrow[u, "d"] \arrow[r, "f"] & B^{q} \arrow[u, "d"] \arrow[r, "g"] & C^{q} \arrow[u, "d"] \arrow[r] & 0 \\
 & {} \arrow[u, "d"] & b \arrow[u, "d"] & c \arrow[u, "d"] &
\end{tikzcd}$$

Let $c \in C^q$, with $dc = 0$. Define

$$\boxed{d^*[c] := [a]}$$

where $[a]$ is obtained as follows (see diagram)

$\boxed{c = g(b)}$ for some $b \in B^q$ (since g is surjective).

Notice that $g(db) \underset{g \text{ chain map}}{=} d\,g(b) = dc = 0$,

that is, $db \in \operatorname{Ker} g$. By exactness ($\operatorname{Ker} g = \operatorname{Im} f$)

$$\boxed{db = f(a)}$$

for a <u>unique</u> $a \in A^{q+1}$ (f being injective).

* We check that $\boxed{da = 0}$ (whence a defines a cohomology class $[a] \in H^{q+1}(A)$). From $d^2 b = 0$ one has:

$$0 = d^2 b = d(db) = d f(a) \underset{f \text{ chain map}}{=} f\,da \underset{\text{injectivity of } f}{\Rightarrow} da = 0$$

⚠ $\boxed{\text{We must prove that } d^* \text{ is well-defined}}$

This will be done in the next lecture