Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

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Lecture XLVI.

HOMOLOGICAL ALGEBRA (CONTINUED)

"short exact sequences of chain complexes induce long exact sequences in cohomology" continued

(2) We have to check that $\underset{dc_1=0}{d^*}$ is well-defined $\underset{da_1=0}{}$

(2.1) we verify that $[c_1]=[c] \Rightarrow [a_1]=[a]$

Let $\underset{g(b_1)}{\overset{}{c_1}} = \underset{g(b)}{c} + d\gamma$. Then $g(b_1)=g(b)+d\gamma$

Let ξ such that $g(\xi)=\gamma$

(g is surjective). Then

$$g(b_1)=g(b)+d\,g(\xi) = g(b)+g(d\xi) = g(b+d\xi)$$

$\Rightarrow \quad b_1 = b + d\xi + x \qquad x \in \operatorname{Ker} g = \operatorname{Im} f$

$\Rightarrow \quad x = f(\eta)$ (for a unique η)

that is

$$b_1 = b + d\xi + f(\eta).$$

But $\underset{f(a_1)}{db_1} = \underset{f(a)}{db} + \underset{0}{d^2\xi} + df(\eta) = db + f\,d\eta$

$\Rightarrow$

$$f(a_1) = f(a) + f\,d\eta = f(a+d\eta)$$

$\underset{\text{injectivity of } f}{\Rightarrow} \quad a_1 = a + d\eta \quad \Rightarrow \quad [a_1]=[a]$

(2.2) Let also check independence of b :

If $\quad g(b_1) = c = g(b)$

then $\quad g(b_1 - b) = 0 \quad \Rightarrow \quad b_1 - b = f(\alpha)$

$$= db_1 - db = d f(\alpha) = f(d\alpha)$$

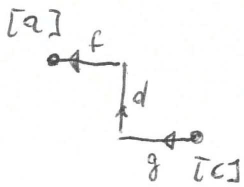
$$\begin{array}{ccc} \| & & \| \\ f(a_1) & - & f(a) \end{array} = f(d\alpha)$$

unique

$$\Rightarrow \quad a_1 = a + d\alpha \qquad \Rightarrow [a_1] = [a]$$

One writes (abuse of notation)

$$[a] = d^* [c] = \Big[(f^{-1} \cdot d \cdot g^{-1}) [c] \Big]_{\text{class}}$$



d^* is a homomorphism (clear.)

(3) We must check that: $\quad d^* g^* = 0 , \qquad g^* f^* = 0 ,$

$$d^* f^* = 0$$

$\boxed{\text{+ thus "Im} \subseteq \text{Ker" in general}}$

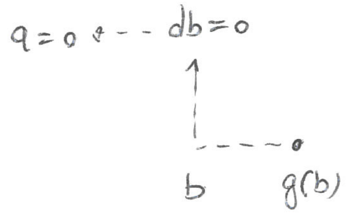
(and exactness at $H^*(A)$, $H^*(B)$, $H^*(C)$.. see step (4))

i.e. the converse: "Ker ⊂ Im"

(3.1) $\quad g^* f^* = (g \circ f)^* = 0^* \qquad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

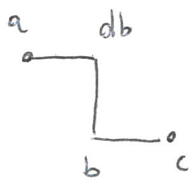
short exact.

(3.2) $\quad \boxed{d^* g^* = 0}$



If $[b] \in H^q(B)$, $db = 0$

$\Rightarrow [a] = 0$



(3.3) $\quad \boxed{f^* d^* = 0}$

$f(a) = db \quad \Rightarrow \quad [f(a)] = 0 \quad$ in $H^{q+1}(B)$

(4) Exactness of the full sequence.

we have already shown in Step 3 that " Im $\subseteq$ Ker " so we are left with proving the reverse inclusions

(4.1) Exactness at $H^*(B)$

Let $[b] \in \ker g_*$ i.e. $g(b) = dc'$

Let b' such that $g(b') = c'$. Then

$$g(b - db') = g(b) - g(db') = dc' - d\,g(b') = dc' - dc' = 0$$

* This shows that we may choose b in such a way that $g(b) = 0$.

Given this, $b = f(a)$ for a unique a, and $da = 0$ $\big(0 = db = df(a) = f(da) \Rightarrow da = 0 \big)$

$\Rightarrow [b] = f^*[a]$

i.e. $[b] \in \operatorname{Im} f^*$

(4.2) Exactness at $H^*(A)$

Let $[a] \in \ker f^*$: $f^*[a] = 0$.

So $f(a) = db$ for some b. Set $c := g(b)$. Then

$dc = d\,g(b) = g(db) = g \cdot f(a) = 0 \Rightarrow [c]$ is defined,

and by construction $d^*[c] = [a]$

$\Rightarrow [a] \in \operatorname{Im} f^*$

$$\begin{tikzcd} a \arrow[r, dash] & db \\ & b \arrow[u, dash] \arrow[r, dash] & c = g(b) \end{tikzcd}$$

(4.3) Exactness at $H^*(C)$

Let $[c]$ such that $d^*[c] = 0$.

One has $a = d\xi$. with the previous notation

$c = g(b)$ $db = f(a)$; furthermore: $d\,f(\xi) = f d\xi = f(a) = db$

$\Rightarrow d(b - f(\xi)) = 0$, and $g(b - f(\xi)) = c$ (since $g f(\xi) = 0$).

But this entails $[c] = g^*[b - f(\xi)]$ This completes the proof. $\square$