V2 — Lectures on **DIFFERENTIAL GEOMETRY AND TOPOLOGY**

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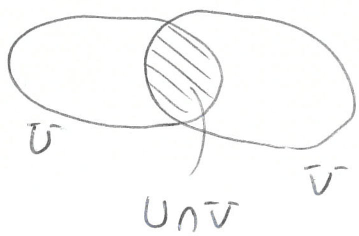
MAYER - VIETORIS

Lecture XLVII

✱ The Mayer-Vietoris sequence for de Rham cohomology.

The MV sequence relates cohomology groups of open sets

$U, V, U\cap V, \underbrace{U\cup V}_{M}$



consider the natural maps

$$M \longleftarrow U \sqcup V \;\underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}}\; U\cap V$$

disjoint union (e.g. $U\times\{0\} \cup V \times \{1\}$)

∂_i : inclusion maps

$$U\cap V \;\underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}}\; \begin{matrix} V \\ U \end{matrix}$$

upon passing to differential forms

(this is an instance of application of a contravariant functor, Λ^* in our case)

pull-back reverses arrows

one gets

$$\Lambda^*(M) \longrightarrow \Lambda^*(U) \oplus \Lambda^*(V) \;\underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}}\; \Lambda^*(U\cap V)$$

recall $\quad \omega|_U := i^*\omega \qquad i : U \hookrightarrow M$

restriction of ω to U $\qquad$ inclusion map

More explicitly:

$$\begin{array}{l}\Lambda^*(M) \ni w \longmapsto w|_{\bar U} \oplus w|_{\bar V} \in \Lambda^*(\bar U)\oplus\Lambda^*(\bar V) \\ \quad\ \ \| \\ U\cap V\end{array}$$

$$\Lambda^*(\bar U) \ni w \xmapsto{\partial_0^*} w|_{\bar V} \in \Lambda^*(U\cap\bar V)$$

$$\Lambda^*(\bar V) \ni w \xmapsto{\partial_1^*} w|_{\bar U} \in \Lambda^*(U\cap V)$$

One arrives at the (short exact) sequence of Mayer-Vietoris

$$\boxed{\begin{array}{ccccccccc}0 & \xrightarrow[\text{inclusion}]{} & \Lambda^*(M) & \xrightarrow{f} & \Lambda^*(\bar U)\oplus\Lambda^*(\bar V) & \xrightarrow{g} & \Lambda^*(U\cap\bar V) & \longrightarrow & 0 \\ & & w & \longmapsto & w|_{\bar U} \oplus w|_{\bar V} & \longrightarrow & w|_{U\cap V} - w|_{U\cap V} = 0 & & \\ & & & & \sigma \oplus \tau & \xmapsto[\partial_0^*-\partial_1^*]{} & \tau-\sigma & & \end{array}}$$

The only subtle point is surjectivity of $g = \partial_0^* - \partial_1^*$..

One has to show that any $\omega \in \Lambda^q(U\cap V)$ is in the image of $\partial_0^* - \partial_1^*$. Let $\{\rho_{\bar U}, \rho_{\bar V}\}$ be an associated partition of unity (to the covering $\{U, \bar V\}$ of $M = U \cup V$)

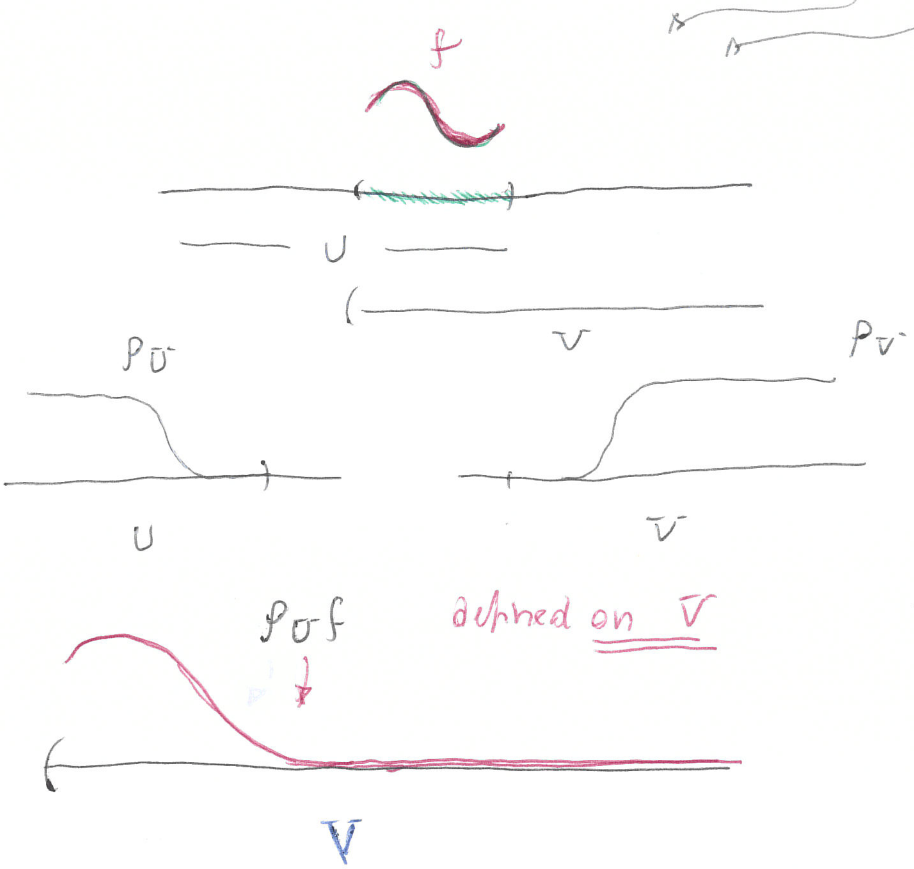
One has:

$$(-\rho_{\bar V} w, \rho_{\bar U} w) \longmapsto w$$

notice this $\rightarrow$ $\underbrace{-\rho_{\bar V} w}_{\in\ \Omega^*(\bar U)}$, $\underbrace{\rho_{\bar U} w}_{\in\ \Omega^*(\bar V)}$

(clear from $\rho_U + \rho_V = 1$)

Let us visualise the case $M = \mathbb{R} = U \cup V$, $f \in \Lambda^0(\mathbb{R})$



Similarly, $\rho_V f$ is defined on U

In view of the general theory, one has a long exact sequence in cohomology (also called MV-sequence)

$$\begin{tikzcd}
H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \\
H^q(M) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[ull, "d^*"'] \\
\cdots \arrow[u, "d^*"] & &
\end{tikzcd}$$

Let us retrace the "zig-zag" construction given in the abstract setting, adapting it to the present context.

Let $\omega \in \Lambda^q(U \cap V)$ be a closed form: $d\omega = 0$ ($[\omega] \in H^q(U \cap V)$). Since the rows are exact, there exists $\xi \in \Lambda^q(U) \oplus \Lambda^q(V)$ which is mapped to ω, e.g.

$$\xi = (-\rho_V \omega , \rho_U \omega)$$

Since all diagrams commute, $d\xi$ is mapped to zero in $\Lambda^{q+1}(U \cap V)$,

$$\begin{tikzcd} d\xi \arrow[r, mapsto] & 0 \\ \xi \arrow[u, "d"] \arrow[r, mapsto] & \omega \arrow[u, "d"'] \end{tikzcd} \quad (+)$$

but this implies that

$$-d(\rho_{\bar V}\,\omega) = d(\rho_{U}\,\omega) \quad \underline{\text{in } U \cap V},$$

hence $d\xi$ comes from some $\alpha \in \Lambda^{q+1}(M = U \cup \bar V)$.

Actually α is <u>closed</u> $(d\alpha = 0)$ and $[\alpha] = d^*[\omega]$. We know from the abstract treatment that this construction is <u>independent of all choices</u>. So, to recap:

$$\boxed{d^*[\omega] = \left[\begin{cases} -d\,\rho_{\bar V}\,\omega & \text{in } U \\ d\,\rho_{\bar U}\,\omega & \text{in } V \end{cases}\right]}$$

class

they agree on $U \cap \bar V$ (+)

Notice that α is <u>manifestly</u> closed (in general, a closed form is locally exact, by the Poincaré lemma)

$$(+) \qquad -d(\rho_V\,\omega) \overset{?}{=} d(\rho_U\,\omega)$$

Let us check this in another way

$$-d\rho_{\bar V} \wedge \omega - \rho_{\bar V}\,\underset{0}{\underset{\parallel}{d\omega}} \overset{?}{=} d\rho_{\bar U} \wedge \omega + \rho_{\bar U}\,\underset{0}{\underset{\parallel}{d\omega}}$$

$$-d\rho_{\bar V} \wedge \omega \overset{?}{=} d\rho_{\bar U} \wedge \omega \qquad \text{YES:} \quad d(\overbrace{\rho_{\bar U} + \rho_V}^{1}) \wedge \omega = 0$$

As a first example, let us compute, in various ways, the cohomology of S^1

$\boxed{H^*(S^1)}$

$H^0(S^1) \cong \mathbb{R}$ ← already proved (or left as an exercise)

$H^1(S^1) \cong \mathbb{R}$

(1) Let w be a 1-form on S^1; then w is automatically closed. Write $w = f(x)dx$, $f(0) = f(2\pi)$;

w is exact if and only if $\int_{S^1} w = 0$, i.e.

$\int_0^{2\pi} f(x)dx = 0$. Indeed, if w is exact,

then $w = dg$ and $\int_{S^1} w = \int_{S^1} dg = \int_0^{2\pi} g'(x)dx =$

$= g(2\pi) - g(0) = 0$ (g must in fact be periodic).

Conversely, if $\int_0^{2\pi} f(x)dx = 0$, then $g(x) :=$

$= \int_0^x f(\varphi)d\varphi \in \Lambda^0(S^1)$ $(g(2\pi) = g(0) = 0)$

and $dg = f(x)dx = w$. Therefore,

the map
$$\begin{aligned} Z^1(S^1) &\longmapsto \mathbb{R} \\ w &\longmapsto \int_{S^1} w \end{aligned}$$

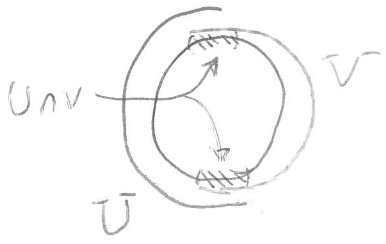
is surjective and its kernel is precisely $B^1(S^1)$,

whence $H^1(S^1) = \dfrac{Z^1(S^1)}{B^1(S^1)} \cong \mathbb{R}$

* The angular form dx generates $H^1(S^1)$.

↑ abuse of notation!

② Let us employ the MV-sequence. Consider the cover shown in figure (notice that $U \cap \bar V$ is not connected). One has



$$S^1 \qquad U \sqcup \bar V \qquad U \cap \bar V$$

$$\begin{tikzcd}
0 \arrow[r] & \boxed{H^0(S^1)} \arrow[r, "f"] & \underset{H^0(U)\oplus H^0(\bar V)}{\mathbb{R}\oplus\mathbb{R}} \arrow[r, "\delta"] & \underset{H^0(U\cap \bar V)}{\mathbb{R}\oplus\mathbb{R}} \arrow[dll, "d^*"] \\
 & \boxed{H^1(S^1)} \arrow[r, "g"] & \overset{H^1(\bar U)\oplus H^1(\bar V)}{0} \arrow[r] & \overset{H^1(U\cap \bar V)}{0}
\end{tikzcd}$$

Poincaré

$*$ $\delta: (\omega, \tau) \longmapsto (\tau - \omega, \tau - \omega)$

notice this

$$H^0(S^1) = \mathrm{Im}\, f = \ker \delta$$

$$\dim \mathrm{Im}\, \delta = 1 \quad \Rightarrow \quad \dim \ker \delta \underset{N+R}{=} 2 - 1 = 1$$

$$\Rightarrow \boxed{H^0(S^1) \cong \mathbb{R}}$$

Furthermore, one has $\mathrm{Im}\, d^* = H^1(S^1) = \ker g$,

$$\ker d^* = \mathrm{Im}\, \delta \underset{N+R}{\Rightarrow} h^1 + 1 = 2 \Rightarrow h^1 = 1,$$

that is $\boxed{H^1(S^1) \cong \mathbb{R}}$

Let us determine another concrete representative for a generator of $H^1(S^1)$, via the MV construction:

take $\alpha \in \Lambda^0(U \cap V)$ which is <u>not</u> in the image of

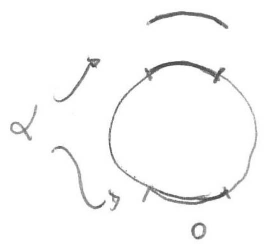
$\delta : \Lambda^0(U) \oplus \Lambda^0(V) \longrightarrow \Lambda^0(U \cap V)$

(otherwise it will be mapped to 0 by d^*), and then

take $d^*\alpha$ recall: $d^*[\omega] = \left[\begin{Bmatrix} -d\rho_V\omega \\ d\rho_U\omega \end{Bmatrix}\right]$

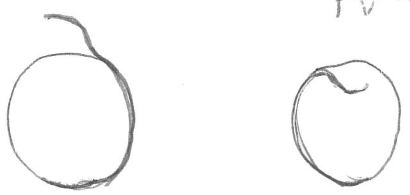
1

α



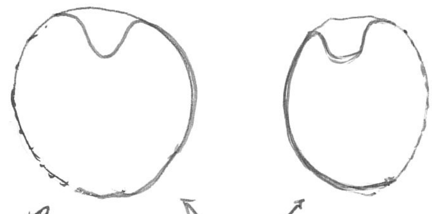
0

$\rho_U\alpha$ $-\rho_V\alpha$

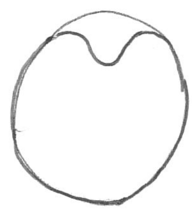


defined on V defined on U

$(-\rho_V\alpha, \rho_U\alpha) \mapsto (\rho_U + \rho_V)\alpha = \alpha$



$\rightsquigarrow$ get a global 1-form β



called <u>bump</u> 1-form

agree on the overlap

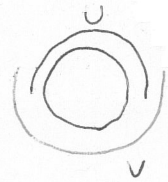
one can then impose the condition

$$\int_{S^1} \beta = 1$$

we are showing the <u>functions</u> <u>multiplying</u> <u>dx</u>

$$\boxed{H^*(S^1) = \begin{cases} \mathbb{R} & q=0 \\ \mathbb{R} & q=1 \end{cases}}$$

Still in another way



$$0 \to \underset{}{H^0(S^1)} \xrightarrow{f} \overset{\mathbb{R}}{H^0(U)} \oplus \overset{\mathbb{R}}{H^0(V)} \xrightarrow{g} \overset{\mathbb{R}^2}{H^0(U\cap V)}$$

$$\xrightarrow{d^*} H^1(S^1) \xrightarrow{h} \overset{0}{H^1(U)} \oplus \overset{0}{H^1(V)} \to \overset{0}{H^1(U \cap V)} \to 0$$

$(\sigma,\tau) \mapsto \tau - \sigma$ ← Recall

$\boxed{\ker f = 0}$ $\quad \boxed{\operatorname{Im} f} = \ker g = \boxed{\mathbb{R}}$

$$\boxed{\begin{aligned} \ker f &= 0 \\ \operatorname{Im} f &= \mathbb{R} \end{aligned}}$$

$\Rightarrow \boxed{H^0(S^1) = \mathbb{R}}$

$$\boxed{\operatorname{Im} d^* = \ker h = H^1(S^1)}$$

$\ker d^* = \operatorname{Im} g \qquad \dim \ker d^* = 1$

$$\underbrace{\dim \operatorname{Im} d^*}_{h^1(S^1)} + \underbrace{\dim \ker d^*}_{1} = 2$$

$$\Rightarrow \boxed{h^1(S^1) = 1}$$

★★ variant: (shorter)

$$h^0(S^1) - (h^0(U) + h^0(V)) + h^0(U\cap V) - \overset{h^1(S^1)}{x} + 0 = 0$$

$$1 - 2 + 2 - x = 0 \quad \Rightarrow \quad x = 1$$