Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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Lecture XXXIX

INTEGRATION

◇ Integration on manifolds

☆ Integration on $\mathbb{R}^n$

In $\mathbb{R}^n$, if $f \in C_c^\infty(\mathbb{R}^n)$ (compact support), one defines the ordinary

Riemann integral

$$\int_{\mathbb{R}^n} f \, dx_1 \dots dx_n$$

If $\omega = f \, dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \in \Lambda_c^n(\mathbb{R}^n)$

(an n-form with compact support), one sets:

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f \, dx_1 \dots dx_n$$

With this definition, one has, for $\omega' = f \, dx_{\pi(1)} \wedge \dots dx_{\pi(n)}$

$\pi \in S_n$ (a permutation),

$$\int_{\mathbb{R}^n} \omega' = \underbrace{(-1)^\pi}_{\text{sign, or parity of } \pi} \int_{\mathbb{R}^n} f \, dx_1 \, dx_n = (-1)^\pi \int_{\mathbb{R}^n} \omega$$

Upon performing a coordinate change (diffeomorphism)

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$y \longrightarrow x$$

$$x = Ty$$
$$x_i \equiv T_i(y_1 \ldots y_n) = x_i \circ T(y_1 \ldots y_n)$$

one has

$$\underset{dT_1 \wedge \ldots dT_n}{dx_1 \wedge \ldots dx_n} = \underbrace{\det\left(\frac{\partial x_i}{\partial y_j}\right)}_{\substack{J(T) \\ \text{Jacobian}}} dy_1 \wedge \ldots dy_n \equiv J(T)\, dy_1 \wedge \ldots dy_n$$

Hence

$$\int_{\mathbb{R}^n} \overset{\text{pull-back}}{T^*}\omega = \int_{\mathbb{R}^n} (f \circ T)\, dT_1 \wedge \ldots dT_n$$

$$= \int_{\mathbb{R}^n} (f \circ T)\, J(T)\, dy_1 \wedge \ldots dy_n$$

However, one also has

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x_1 \ldots x_n)\, dx_1 \wedge \ldots \wedge dx_n$$
$$= \int_{\mathbb{R}^n} f(x_1 \ldots x_n)\, dx_1 \ldots dx_n$$
$$= \int_{\mathbb{R}^n} (f \circ T)\, |J(T)|\, dy_1 \ldots dy_n$$

notice this

whence

$$\boxed{\int_{\mathbb{R}^n} T^*\omega = \pm \int_{\mathbb{R}^n} \omega}$$

T is said to be orientation preserving if $J(T) > 0$

otherwise ($J(T) < 0$) T is called orientation reversing.

Recall that two bases b, b' in $\mathbb{R}^n$ are called equioriented (or: yield the same orientation) if $\det \underbrace{m_{bb'}(I)}_{\text{change of basis}} > 0$

We also write

$$T^* dx_1 \wedge \dots dx_n = J(T)\, dy_1 \wedge \dots dy_n$$

$$\text{(upon changing names)} \qquad \equiv \quad \underset{\overset{\|}{J(T)}}{\alpha(x)}\, dx_1 \wedge \dots dx_n$$

$T : \mathbb{R}^n \to \mathbb{R}^n$ is then orientation preserving if and only if $\alpha > 0$.

★ Orientable manifolds

Let M be a smooth manifold, $\mathcal{A} = \{ \mathcal{U}_\alpha, \varphi_\alpha \}_{\alpha \in \Omega}$ an atlas for M. $\mathcal{A}$ is said to be oriented if the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are orientation preserving. If M possesses an oriented atlas, then it is called orientable.

The following is important:

* **Proposition** A smooth manifold M, $\dim M = n$ is orientable if and only if there exists a never vanishing global n-form ω on M ($\omega \in \Lambda^n(M)$, $\omega \neq 0$ at every point).

Proof.

($\Leftarrow$) Let ω be a global n-form, $\omega(x) \neq 0 \ \forall x \in M$.

Given a local chart φ_α, one has

x_i : local coordinates in $\mathbb{R}^n$

$$\varphi_\alpha^* dx_1 \wedge \dots dx_n = f_\alpha \omega \quad , \quad f_\alpha \neq 0 \text{ on } \mathcal{U}_\alpha$$

similarly $\varphi_\beta^* dx_1 \wedge \dots dx_n = f_\beta \omega \qquad f_\beta \neq 0$ on $\mathcal{U}_\beta$

One can arrange things (by a suitable switch of coordinates) in such a way that $f_\alpha > 0 \quad \forall \alpha \in \Omega$

Then we have: $\omega = f_\alpha^{-1} \cdot \varphi_\alpha^* (dx_1 \wedge \dots dx_n) = f_\beta^{-1} \varphi_\beta^* (dx_1 \wedge \dots dx_n)$

$\Rightarrow \varphi_\alpha^* (dx_1 \wedge \dots dx_n) = \underbrace{f_\alpha f_\beta^{-1}}_{f} \varphi_\beta^* (dx_1 \wedge \dots dx_n)$. Then

$$(\varphi_\beta^{-1})^* \varphi_\alpha (dx_1 \wedge \dots dx_n) = (\varphi_\beta^{-1})^* [f \varphi_\beta^* (dx_1 \wedge \dots dx_n)] = \text{(functoriality)}$$

$$(\varphi_\beta^{-1*} f) \underbrace{(\varphi_\beta^{-1})^* \varphi_\beta^*}_{= \text{identity}} (dx_1 \wedge \dots dx_n) = \underbrace{(f \circ \varphi_\beta^{-1})(x)}_{= \lambda(x) > 0} dx_1 \wedge \dots dx_n$$

which shows that M is oriented

($\Rightarrow$) Conversely, given M oriented, we have:

$$(\varphi_\alpha \circ \varphi_\beta^{-1})^* dx_1 \wedge \dots \wedge dx_n = \underbrace{\lambda(x)}_{> 0} dx_1 \wedge \dots dx_n$$

$$\|$$

$$(\varphi_\beta^{-1})^* \varphi_\alpha dx_1 \wedge \dots \wedge dx_n$$

$$\|$$

$$(\varphi_\beta^*)^{-1} \varphi_\alpha^* dx_1 \wedge \dots dx_n$$

$\Rightarrow \overbrace{\varphi_\alpha^* (dx_1 \wedge \dots dx_n)}^{\omega_\alpha}$

$= \varphi_\beta^* (\lambda \, dx_1 \wedge \dots \wedge dx_n)$

$= \underbrace{(\lambda \circ \varphi_\beta)}_{f > 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta} \underbrace{\varphi_\beta^* (dx_1 \wedge \dots dx_n)}_{\omega_\beta}$

That is

$$\boxed{\omega_\alpha = f\, \omega_\beta} \qquad \text{on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta,$$

and $\underline{\underline{f > 0}}$ thereon

★ Given a partition of unity $\{\rho_\alpha\}_{\alpha\in\Omega}$ subordinate to the covering $\{\mathcal{U}_\alpha\}_{\alpha\in\Omega}$, set

$$\boxed{\omega = \sum_{\alpha\in\Omega} \rho_\alpha\, \omega_\alpha}$$

Since $\rho_\alpha \geq 0$, $\sum_\alpha \rho_\alpha \equiv 1$, it follows that

$\omega \neq 0$ everywhere. □

If M is orientable (assume M connected as well)

then, given ω, ω' never vanishing n-forms on M, one has $\omega' = f\omega$, $f \neq 0$ everywhere

Actually $f \gtrless 0$.

Introduce the following equivalence relation on such forms: $\omega \sim \omega'$ if $\omega' = f\omega$, $f > 0$

Then $[\omega] = \{\omega' \,/\, \omega' = f\omega,\ f > 0\}$ (the class of ω)

Actually, one has exactly two equivalence classes: each of these is called an orientation on M.

An oriented manifold is an orientable manifold M together with the choice of an orientation (notation: $[M]$)

Example: $\mathbb{R}^n$, with $[\omega = dx_1 \wedge \ldots \wedge dx_n] = \{f dx_1 \wedge .. dx_n, \ f>0\}$ is an oriented manifold (this is the standard orientation).

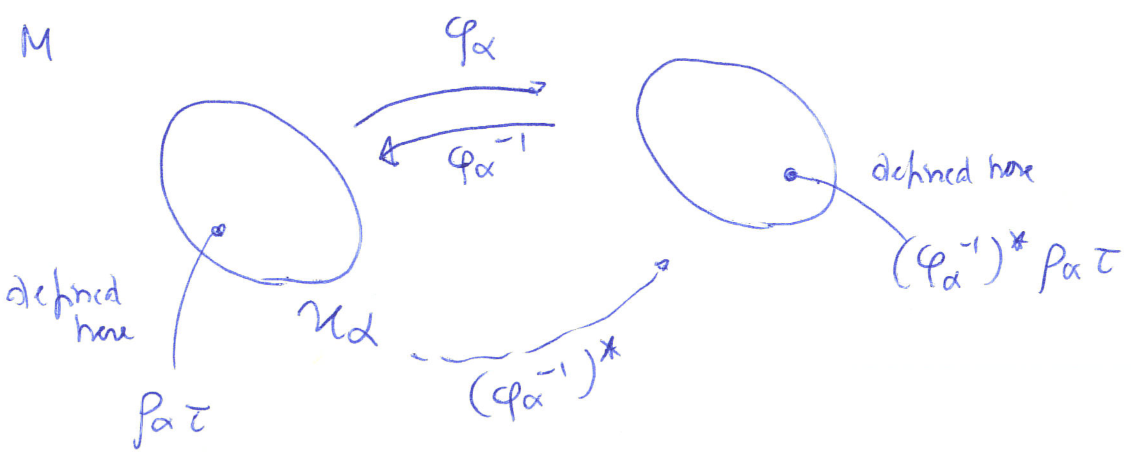
* Integration of n-forms on an oriented manifold

Now, consider an oriented manifold $[M]$ (very often we shall discard $[\ ]$).

Let $\tau \in \Lambda^n_c(M)$. Define

$$\int_{[M]} \tau := \sum_{\alpha\in\Omega} \int_{\mathcal{U}_\alpha} \rho_\alpha \cdot \tau := \sum_{\alpha\in\Omega} \int_{\mathbb{R}^n} (\varphi_\alpha^{-1})^* \cdot \rho_\alpha \tau$$

(of course $\mathcal{A} = \{\mathcal{U}_\alpha, \varphi_\alpha\}_{\alpha\in\Omega}$ is an oriented atlas, and $\{\rho_\alpha\}_{\alpha\in\Omega}$ a partition of unity subordinate to $\{\mathcal{U}_\alpha\}_{\alpha\in\Omega}$



(* $\int_{\mathcal{U}_\alpha} \rho_\alpha \tau$ is well-defined, in view of orientability)

We write, for brevity

$\int_M \tau$ instead of $\int_{[M]} \tau$

Of course, we have to check that $\int_M \tau$ is <u>well-defined</u>, i.e. it depends neither on $\mathcal{A}$ (oriented atlas), nor on $\{\rho_\alpha\}$. This will be done presently.

First of all, observe that

$$\underset{\text{closed}}{\operatorname{supp} \rho_\alpha \tau} \subseteq \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \tau_\alpha \subset \underset{\text{compact}}{\operatorname{supp} \tau}$$

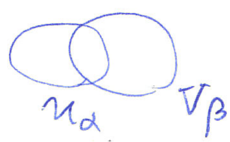
But, in a <u>Hausdorff</u> space, K <u>closed</u> $\subseteq$ H <u>compact</u> $\Rightarrow$ K <u>compact</u>, so $\rho_\alpha \tau$ (and $(\varphi_\alpha^{-1})^* \rho_\alpha \tau$) has compact support.

Let $\mathcal{B} = \{V_\beta, \psi_\beta\}_{\beta \in B}$ be another oriented atlas (compatible with the former), and let $\{\chi_\beta\}_{\beta \in B}$ be an associated partition of unity. Then:

$$\boxed{\sum_\alpha \int_{U_\alpha} \rho_\alpha \tau} = \sum_{\alpha, \beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \tau \qquad \left(\text{since } \sum_\beta \chi_\beta = 1\right)$$

Now $\operatorname{supp} \rho_\alpha \chi_\beta \subseteq U_\alpha \cap V_\beta$

may be empty



$$\Rightarrow \sum_{\alpha, \beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \tau = \sum_{\alpha, \beta} \int_{V_\beta} \rho_\alpha \chi_\beta \tau$$

$$\underset{\left(\text{since } \sum_\alpha \rho_\alpha = 1\right)}{=} \sum_\beta \boxed{\int_{V_\beta} \chi_\beta \tau} \quad \square$$