Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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Lecture XXXVIII

POINCARÉ LEMMA - NON TRIVIALITY OF SOME COHOMOLOGY GROUPS

Let us resume our discussion on de Rham theory

★ Poincaré lemma

$$\boxed{H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ \{0\}, & k > 0 \end{cases}} \qquad n \geq 1$$

Proof. The case $k = 0$ is easily dealt with: if $df = 0 \Rightarrow f = c$ (constant) since $\mathbb{R}^n$ is connected. There are no exact 0-forms (functions), and the conclusion follows.

In general, we construct a sequence of linear maps

$$h_j : \Lambda^{j+1} \longrightarrow \Lambda^j$$

such that

$$(\ast) \quad \boxed{h_k \circ d + d \circ h_{k-1} = \underset{\substack{\uparrow \\ \text{identity}}}{id_{\Lambda^k}}}$$

The collection $\{h_j\}$ is called "homotopy operator"

$$\cdots \longrightarrow \Lambda^{k-1} \xrightarrow{d} \Lambda^k \xrightarrow{d} \Lambda^{k+1} \longrightarrow \cdots$$

$$\Lambda^{k-1} \xleftarrow{h_{k-1}} \Lambda^k \xleftarrow{h_k} \Lambda^{k+1}$$

Formulae like $(\ast)$ are extremely important in algebraic topology.

Given $\{h_j\}$, one would find, for $\omega \in Z^k$

$$h_k \underbrace{d\omega}_{\substack{\| \\ 0}} + d \underbrace{h_{k-1}\omega}_{\in \Lambda^{k-1}} = \omega$$

We also observe in advance that the same result and method of proof will hold for star-shaped open sets

i.e. $\omega = d(h_{k-1}\omega) \Rightarrow \omega$ would be exact.

By linearity, it suffices to define h_{k-1} (say) on $\omega = g\, dx^{i_1} \wedge \dots dx^{i_k}$. So let

$$h_{k-1}(\omega)(\underset{(x_1 \dots x_n)}{\underset{\shortparallel}{x}}) := \left(\int_0^1 t^{k-1} g(tx)\, dt\right) \mu ,$$

where

related to $dx^{i_1} \wedge \dots dx^{i_k}$

$$\mu = \sum_{j=1}^{k} (-1)^{j-1} x^{i_j} dx^{i_1} \wedge \dots \widehat{dx^{i_j}} \wedge \dots dx^{i_k}$$

observe that

$j-1$ switches

$$d\mu = \sum_{j=1}^{k} (-1)^{j-1} dx^{i_j} \wedge dx^{i_1} \wedge \dots \widehat{dx^{i_j}} \wedge \dots dx^{i_k}$$

$$= \sum_{j=1}^{k} dx^{i_1} \wedge \dots dx^{i_k} = k\, dx^{i_1} \wedge \dots dx^{i_k}$$

example: $\mu = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2$

$$d\mu = dx_1 \wedge dx_2 \wedge dx_3 - dx_2 \wedge dx_1 \wedge dx_3 + dx_3 \wedge dx_1 \wedge dx_2$$
$$= dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge dx_3$$
$$= 3\, dx_1 \wedge dx_2 \wedge dx_3$$

also consider $dx_4 \wedge \frac{1}{3} d\mu = dx_4 \wedge (dx_1 \wedge dx_2 \wedge dx_3)$ (say).

then μ' (relative to $\uparrow$) is

$$\mu' = x_4 \wedge dx_1 \wedge dx_2 \wedge dx_3 - x_1\, dx_4 \wedge dx_2 \wedge dx_3 + x_2\, dx_4 \wedge dx_1 \wedge dx_3$$
$$- x_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 = x_4 (dx_1 \wedge dx_2 \wedge dx_3) - dx_4 \wedge \mu$$

This will be needed soon

① Let us compute

$$\boxed{d\left(h_{k-1}(\omega)\right)(x)} = d\left[\left(\int_0^1 t^{k-1} g(tx)\,dt\right)\mu\right] =$$

$$= d\left(\int_0^1 t^{k-1} g(tx)\,dt\right) \wedge \mu + \left\{\int_0^1 t^{k-1} g(tx)\,dt\right\} \cdot d\mu$$

$$= \sum_{j=1}^{n} \left\{\int_0^1 \overbrace{t^{k-1} \cdot t}^{t^k} \frac{\partial g}{\partial x_j}(tx)\,dt\right\} dx^j \wedge \mu$$

$$+ \left(\int_0^1 t^{k-1} g(tx)\,dt\right) \cdot k\, dx^{i_1} \dots dx^{i_k}$$

$$\boxed{= \sum_{j=1}^{n} \left(\int_0^1 t^k \frac{\partial g}{\partial x_j}(tx)\,dt\right) dx^j \wedge \mu + k \cdot \left(\int_0^1 t^{k-1} g(tx)\,dt\right) dx^{i_1} \wedge \dots \wedge dx^{i_k}}$$

② Now compute

$$h_k(d\omega)(x) = h_k\left[\sum_{j=1}^{n} \frac{\partial g}{\partial x_j}\, dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}\right]$$

$$= \sum_{j=1}^{n} \left(\int_0^1 t^k \frac{\partial g}{\partial x_j}(tx)\,dt\right) \left[\overbrace{x^j\, dx^{i_1} \wedge \dots dx^{i_k}}^{\mu'} - dx^j \wedge \mu\right]$$

cf. the calculation in the preceding page

Then

$$\textcircled{1} + \textcircled{2} = (d \circ h_{k-1} + h_k \circ d)(\omega)(x) =$$

$$\sum_{j=1}^{n} \left[\int_0^1 t^k \frac{\partial g}{\partial x_j}(tx)\,dt\right] \left\{ \cancel{dx^j \wedge \mu} + x^j dx^{i_1} \wedge \ldots dx^{i_k} - \cancel{dx^j \wedge \mu} \right\}$$

$$+ \left(k \int_0^1 t^{k-1} g(tx)\,dt \right) dx^{i_1} \wedge \ldots dx^{i_k}$$

$$= \left(\int_0^1 t^k \overbrace{\sum_{j=1}^{n} x_j \frac{\partial g}{\partial x_j}(tx)}^{\frac{dg}{dt}(tx)} dt \right) dx^{i_1} \wedge \ldots dx^{i_k}$$

$$+ \int_0^1 \underbrace{k t^{k-1}}_{\frac{dt^k}{dt}} g(tx)\,dt \,) \, dx^{i_1} \wedge \ldots dx^{i_k}$$

$$= \left(\int_0^1 \frac{d(t^k g)}{dt}\,dt \right) dx^{i_1} \wedge \ldots dx^{i_k}$$

$$= \left(t^k g(tx) \Big|_0^1 \right) dx^{i_1} \wedge \ldots dx^{i_k}$$

$$= g(x)\, dx^{i_1} \wedge \ldots dx^{i_k} = \omega$$

$\square$

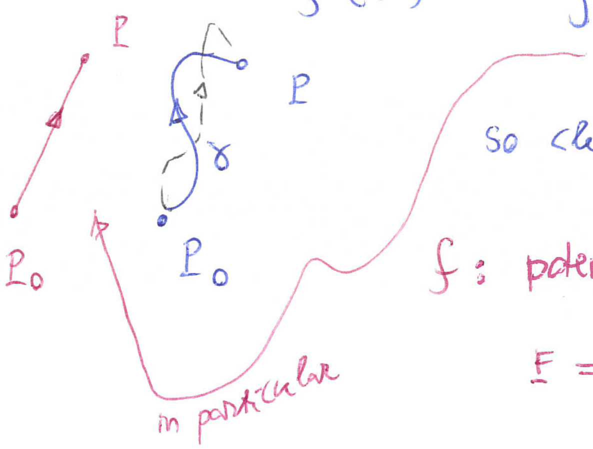
Comment on the definition of $\{h_k\}$

For a closed 1-form (on $\mathbb{R}^2$, say) $\omega = a\,dx + b\,dy$

$\left(d\omega = 0 \iff \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} = 0 \right)$, exactness is

proved by taking

$$f(P) = \int_\gamma \omega$$ (this is well defined - use Green's theorem)



so clearly $df = \omega$

f: potential for the conservative force

$\underline{F} = a\,\underline{i} + b\,\underline{j}$, $\omega = \underline{F} \cdot d\underline{r}$

work 1-form

The construction of $\{h_k\}$ generalizes precisely this idea.

The de Rham group are non-trivial in general and actually detect topological properties of the underlying manifold. Let us discuss some simple examples, coming from physics.

1. $\mathbb{R}^2 \setminus \{(0,0)\}$ Consider the "angular form"

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

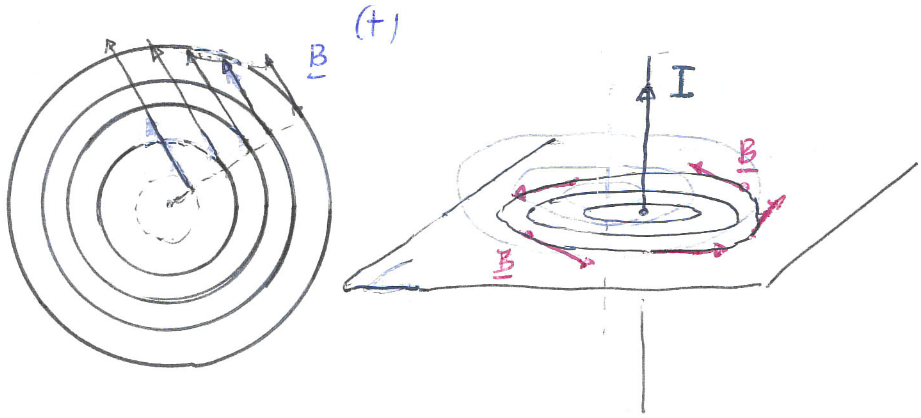
It may be written as "mixed formalism"

$$\omega = \frac{i\underline{r}\cdot d\underline{r}}{\|\underline{r}\|^2} \equiv \underline{B}\cdot d\underline{l} \qquad \underline{r} = (x,y),\quad d\underline{r} = (dx, dy)$$

physically, ω is the element of circulation of the magnetic field determined by a rectilinear wire wherein an electric current flows (Biot-Savart law)

$i \equiv$ counterclockwise rotation by $\frac{\pi}{2}$

$\|\underline{B}\| \sim \frac{1}{r}$



(+) or $\underline{\nu}$, see below

1'.| In fluid mechanics, $\underline{B} \equiv \underline{v}$ is

the velocity field of an irrotational perfect fluid

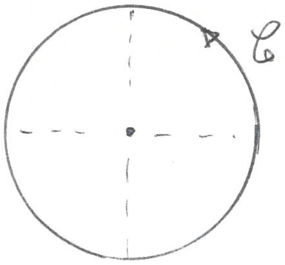
($d\omega = 0 \iff \text{curl}\, \underline{v} = \underline{0}$) on $\mathbb{R}^2 \setminus \{(0,0)\}$

notice that $\text{div}\, \underline{v} = 0$)

possessing a vortex at the origin (same picture)

☆ Let us then show that $\boxed{H^1(\mathbb{R}^2 \setminus \{(0,0)\}) \text{ is non trivial}}$

(later on we shall see that indeed this group is $\mathbb{R}$)



Take, for instance, the unit circle $x^2 + y^2 = 1$, oriented counterclockwise

Then an easy calculation shows that

$$\int_{\mathcal{C}} \omega = 2\pi$$

But, if ω were exact, $\omega = df$, then we would have had, instead:

$$\int_{\mathcal{C}} \omega = \int_{\mathcal{C}} df = 0$$

i.e. a contradiction.

2. Let us now check that

$$\boxed{H^2(\mathbb{R}^3 \setminus \{(0,0,0)\}) \text{ is non trivial}}$$

(indeed, it is $\mathbb{R}$)

("Gauss' law" $\operatorname{div} \underline{E} = \rho$) in electrostatics

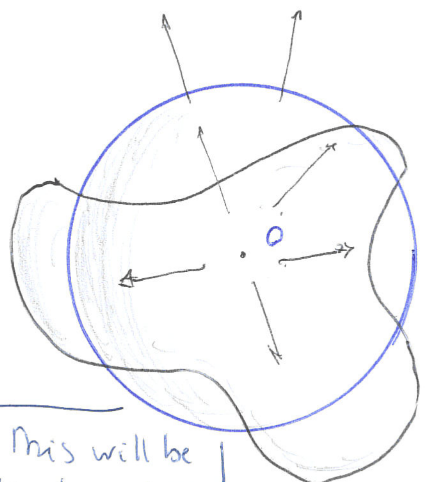
Let E be the flux 2-form of $\underline{E} = \dfrac{\underline{r}}{\|\underline{r}\|^3}$

electric field generated by a point charge located at the origin

i.e.

$$E = \frac{x}{(x^2+y^2+z^2)^{3/2}}\, dy \wedge dz + \frac{y}{(x^2+y^2+z^2)^{3/2}}\, dz \wedge dx + \frac{z}{(x^2+y^2+z^2)^{3/2}}\, dx \wedge dy$$

$(E = \underline{E} \cdot d\underline{\sigma} = \underline{E} \cdot \underline{n}\, d\sigma)$



One has $dE = 0$

(i.e. $\operatorname{div} \underline{E} = 0$: $\underline{E}$ is "solenoidal")

One has $\boxed{\int_\Sigma E = \iint_\Sigma \underline{E} \cdot \underline{n}\, d\sigma = 4\pi}$

take e.g. a sphere centred at 0

integral of a 2-form, we shall formalise it later on

Σ = closed oriented surface surrounding 0

All this will be better formalised later on

If E were exact, then

$E = da$, or $\underline{E} = \operatorname{curl} \underline{A}$

But $\iint_{\Sigma'} \operatorname{curl} \underline{A} \cdot \underline{n}\, d\sigma =$ (Stokes)

$\int_{\mathscr{C}} \underline{A} \cdot d\underline{r}$, which goes to zero as $\mathscr{C}$ shrinks to a point.