Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

VI2

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Lecture XLIII

POINCARÉ LEMMA FOR SINGULAR COHOMOLOGY

SIMPLICIAL HOMOLOGY

✱ The Poincaré lemma for singular homology

(cone construction)

$$\boxed{H_q(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0 & q \geq 1 \end{cases}}$$

we construct a homotopy operator K.

(cf. the Poincaré lemma for de Rham cohomology)

Let s be a q-simplex in $\mathbb{R}^n$

$$s : \Delta_q \to \mathbb{R}^n$$

Define the "cone over s"

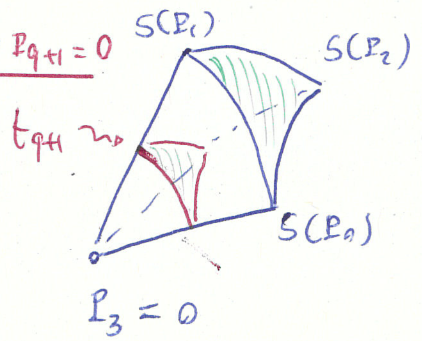
$P_{q+1} = 0$ $\qquad$ "time" $\qquad$ $\sum_{j=0}^{q} t_j = 1 - t_{q+1}$

$$\boxed{(Ks)\Big(\sum_{j=0}^{q+1} t_j P_j\Big) := (1 - t_{q+1})\, s\Big(\sum_{j=0}^{q} \underbrace{\frac{t_j}{(1-t_{q+1})}}_{\text{they should sum to } 1} P_j\Big)}$$

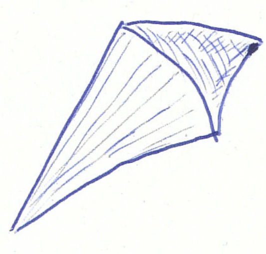
vertices of s

$t_j \geq 0 \quad \sum t_j = 1$

cone with base s and vertex $P_{q+1} = 0$



⚠ Notice that Ks does not preserve smoothness of simplices, however, one also has $H_q^{C^\infty}(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & q=0 \\ 0, & q \geq 1 \end{cases}$



One finds

$$\boxed{\partial K - K\partial = (-1)^{q+1}\,\mathrm{id}} \qquad q \geq 1$$

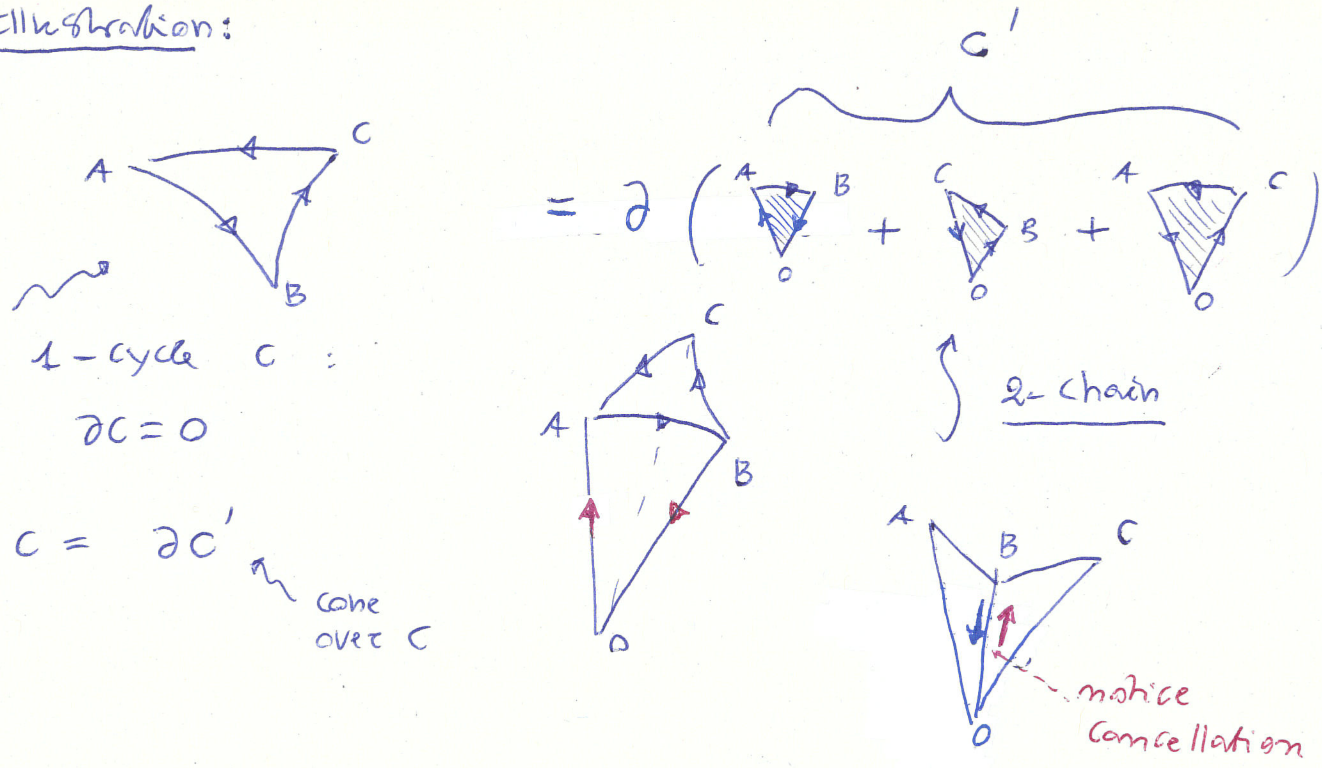
$\Rightarrow$ p-cycles ($p \geq 1$) are boundaries:

c p-cycle : $\partial c = 0$

$$\partial K c - K \underset{\overset{\shortparallel}{0}}{\partial c} = (-1)^{p+1} c$$

$$\Rightarrow \quad c = \partial \left\{ (-1)^{p+1} K c \right\}$$

<u>Illustration:</u>

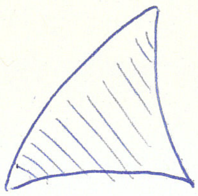


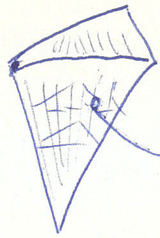
A 1-cycle c :

$\partial c = 0$

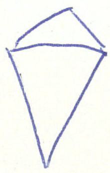
$c = \partial c'$ cone over c

Another example (ignore sign subtleties, for the moment)

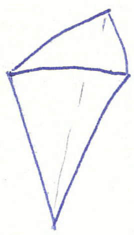
s 

Ks  "full" tetrahedron

s: 2-simplex

$\partial(Ks) =$  "empty" tetrahedron (faces only)

$\partial s =$  "empty" triangle

$K \partial s =$  "empty" tetrahedron, without  s itself

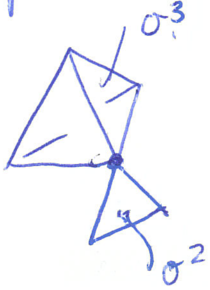
$$\partial(Ks) - (K \partial s) = (-1)^{q+1} s$$

* Simplicial complexes and their homology.

We make a digression on simplicial homology (historical and pedagogical reasons)

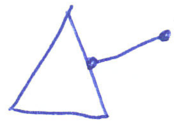
Two simplices are said to be **properly joined** if their intersection is either empty or a face$^{(+)}$ of both simplices

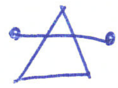
((+) a simplex is a face of another if its vertices belong to the latter)



← properly joined

NO





NO

Def. (i) A geometric complex K (or simplicial complex) is a finite union of simplices which are properly joined and such every face of a simplex in K is itself a face in K

(ii) The dimension of K is by definition the biggest $r \in \mathbb{N}$ such that there exists an r-simplex in K.
(notation: $\dim K$)

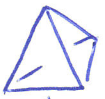
(iii) K, equipped with the relative topology inherited from the ambient space, is called geometric carrier, or polyhedron associated to K; notation: $|K|$, but we actually will blur the distinction between geometric complexes and polyhedra

(iv) If a topological space X is homeomorphic to some $|K|$ ($X \approx |K|$) we say it is triangulable then $|K|$ is, in truth, a triangulation of X

Further notions:

<u>n-pseudomanifold</u>: A polyhedron K such that

a) every simplex in K is a <u>face</u> of an n-simplex in K



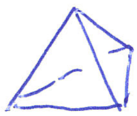
b) every (n-1)-simplex is a face of <u>exactly two</u> n-simplices

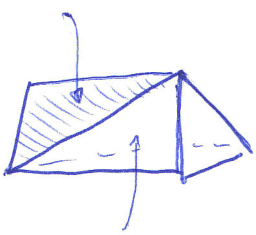
c) given two n-simplices, there exists a sequence on n-simplices beginning with the first one ending with the second one such that the successive intersections yield a (n-1)-face

<u>n-pseudomanifold with boundary</u>

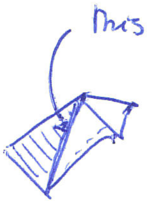
(b) is replaced by

(b') every (n-1)-simplex is a face of <u>at least one</u> and <u>at most two</u> n-simplices





← This is a geometric complex but <u>not</u> a 2-pseudomanifold

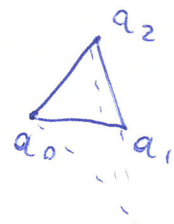


: Still not a pseudomanifold: the fin is not a face of a 3-symplex.

4 Orientation

Given a simplex, an orientation thereof is an ordering of its vertices, up to parity.

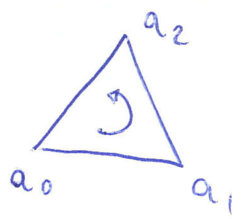
Example. $\langle a_0, a_1 \dots a_p \rangle$ $\rightsquigarrow$ p-simplex with vertices $a_0, a_1 \dots a_p$ (l. ind. if we actually want it to be p-dimensional)



$+\langle a_0, a_1, \dots a_p \rangle$

natural orientation

up to even permutations



$(0,1,2)$
$(1,2,0)$
$(2,0,1)$

$-\langle a_0, a_1, \dots a_p \rangle$ opposite orientation (odd permutation)

The notation $\pm \langle a_0, a \dots a_p \rangle$ is consistent with the concept of p-chain $c = \sum_{\sigma} \underset{\in \mathbb{Z}}{n_\sigma} \cdot \underset{p\text{-simplex}}{\sigma}$

(notation: C_p) : again one manufactures a (finitely generated, in this case) abelian group.

An orientation on a p-simplex induces an orientation on its $(p-1)$-faces, which may differ from the orientation that may have been already assigned to the latter.

The ensuing combinatorics is governed by the incidence numbers

$$[\sigma^{p+1}, \sigma^{p}] := \begin{cases} 0 & \text{if } \sigma^p \text{ is not a face of } \sigma^{p+1} \\ \pm 1 & \text{orientations compatible or not} \end{cases}$$

σ^{p+1}: $(p+1)$-simplex; σ^{p}: p-simplex

The incidence numbers give rise to incidence matrices

$$\eta(p) = \left(\eta_{ij}(p) = [\sigma_i^{p+1}, \sigma_j^{p}] \right)$$

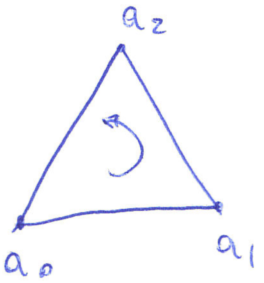
$\eta(p)$: p-th incidence matrix

i: rows, j: columns

$m_1 \times m_2$

m_1: $(p+1)$-simplices, m_2: p-simplices

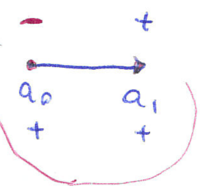
Example



$$\eta(1) = (1, 1, -1)$$

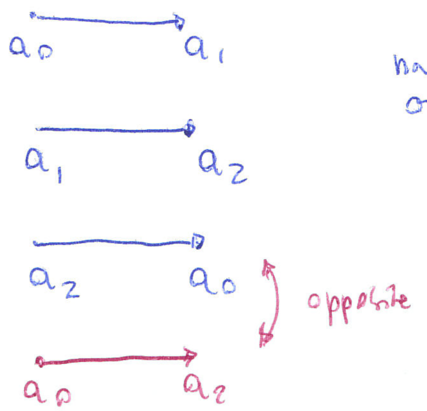
1 2-simplex
3 1-simplices

$$\eta(0) = \begin{pmatrix} -1 & +1 & 0 \\ 0 & -1 & 1 \\ +1 & 0 & -1 \end{pmatrix}$$



3 1-simplices
3 0-simplices

natural orientation on 0-simplices
$+\langle a_i \rangle$



The basic lemma is the following

$$(\star) \quad \sum_{\sigma^{p-1}} [\sigma^p, \sigma^{p-1}][\sigma^{p-1}, \sigma^{p-2}] = 0$$

and leads to the following definition / theorem

$$\partial_p : C_p \longrightarrow C_{p-1}$$

$\star$ boundary map

$$c = \sum n_i \sigma_i^p \longmapsto \partial_p\left(\sum_i n_i \sigma_i^p\right)$$

$$= \sum_i n_i (\partial_p \sigma_i^p)$$

$$:= \sum_i n_i \sum_j [\sigma_i^p, \sigma_j^{p-1}] \sigma_j^{p-1}$$

⚠ it is consistent with the definition of ∂ in singular homology

and one has $(\star\star)$ $\boxed{\partial_{p-1} \circ \partial_p = 0}$

(coming from $(\star)$)

$\leadsto$ one defines simplicial homology groups via the following steps.

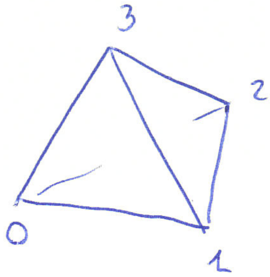
$$Z_p(K) = \{ c \in C_p \,/\, \partial c = 0 \} \quad \text{p-cycles}$$

$$B_p(K) = \{ b \in C_p \,/\, b = \partial c',\ c' \in C_{p+1} \}$$

p-boundaries

$$\boxed{H_p(K) := Z_p(K) / B_p(K)}$$

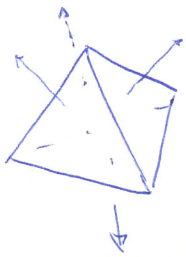
Preliminary examples



$\langle 0,1,2,3 \rangle$ 3-simplex

$$\partial(\langle 0,1,2,3\rangle) = \underbrace{\langle 1,2,3\rangle}_{0\text{ omitted}} - \underbrace{\langle 0,2,3\rangle}_{1\text{ omitted}} + \underbrace{\langle 0,1,3\rangle}_{2\text{ omitted}} - \underbrace{\langle 0,1,2\rangle}_{3\text{ omitted}}$$

(also cf. the def. in singular homology) !

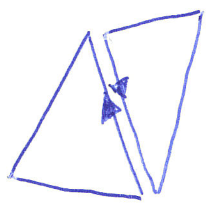


||| this is the empty tetrahedron, oriented through the outer normal (informal language)

Let us check in this case that $\partial^2 = 0$:

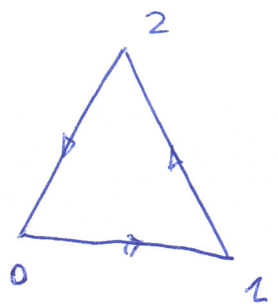
$$\partial\left(\langle 1,2,3\rangle - \langle 0,2,3\rangle + \langle 0,1,3\rangle - \langle 0,1,2\rangle\right)$$

$$= \langle 2,3\rangle - \langle 1,3\rangle + \langle 1,2\rangle -$$
$$-\left(\langle 2,3\rangle - \langle 0,3\rangle + \langle 0,2\rangle\right)$$
$$+\left(\langle 1,3\rangle - \langle 0,3\rangle + \langle 0,1\rangle\right)$$
$$-\left(\langle 1,2\rangle - \langle 0,2\rangle + \langle 0,1\rangle\right)$$



$$= \langle 2,3\rangle - \langle 1,3\rangle + \langle 1,2\rangle$$
$$- \langle 2,3\rangle + \langle 0,3\rangle - \langle 0,2\rangle$$
$$+ \langle 1,3\rangle - \langle 0,3\rangle + \langle 0,1\rangle$$
$$- \langle 1,2\rangle + \langle 0,2\rangle - \langle 0,1\rangle = 0$$

This is the geometric phenomenon: each 1-simplex appears twice, with opposite orientations



$$C = \langle 0,1\rangle + \langle 1,2\rangle + \langle 2,0\rangle$$
$$\in Z_1(K = \triangle)$$
↑ "empty"

indeed: $\partial C = \langle 1\rangle - \langle 0\rangle + \langle 2\rangle - \langle 1\rangle + \langle 0\rangle - \langle 2\rangle$
$= 0$

in this case C is <u>not</u> a boundary (there are no 2-chains).

But if $K = \blacktriangle$, then $\partial C = 0$, and
$$C = \partial\sigma = \partial(\langle 0,1,2\rangle)$$

Let $C = n_0\langle 0\rangle + n_1\langle 1\rangle + n_2\langle 2\rangle$ be a 0-chain. C is also a 0-cycle ($\partial\langle 0\rangle = 0$ by def. etc)

We claim that the boundaries are sums of chains of the form $C = n(\langle 0\rangle - \langle 1\rangle)$ $n \in \mathbb{Z}$ etc

In fact let $C' = n'_1\langle 0,1\rangle + n'_2\langle 1,2\rangle + n'_3\langle 2,0\rangle$

$$\partial C' = n'_1(\langle 1\rangle - \langle 0\rangle) + n'_2(\langle 2\rangle - \langle 1\rangle) + n'_3(\langle 0\rangle - \langle 2\rangle).$$

We compute the <u>homology groups</u> of $K = \triangle$

$H_0(K)$ Notice that $\langle 0\rangle \sim \langle 1\rangle$ etc.

i.e $\underbrace{\langle 1\rangle - \langle 0\rangle}_{\text{0-cycles}} = \partial(\underbrace{\langle 0,1\rangle}_{\text{1-chain}})$