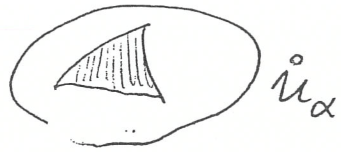
Let $\mathcal{U}$ be a collection of subsets of X whose interiors cover X.

Let $\Delta_*^{\mathcal{U}}(X) \overset{i}{\hookrightarrow} \Delta_*(X)$ ← singular chains

be the subcomplex generated by those singular simplices which are $\mathcal{U}$-small : $\operatorname{Im}\sigma \subset \mathring{\mathcal{U}}_\alpha$

$\alpha = \alpha(\sigma)$



Fact

The (natural) inclusion map induces an isomorphism in homology

$$\boxed{i_*: H_*^{\mathcal{U}}(X) \to H_*(X)}$$

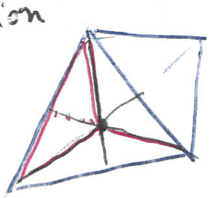
$$\| $$

$$(H_*(\Delta_*^{\mathcal{U}}(X)))$$

(Key idea : successive subdivisions of simplices)

cf: barycentric subdivision

see e.g. Bredon



SINGULAR COHOMOLOGY

DE RHAM'S THEOREM

V2 DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture LVII

Theorem. Let $\mathcal{U}$ be a collection of subsets of X whose interiors cover X.

Set $\Delta^*_{\mathcal{U}}(X; G) = \mathrm{Hom}\,(\Delta^{\mathcal{U}}_*(X), G)$

The maps

$$\Delta^*(X; G) \to \Delta^*_{\mathcal{U}}(X; G)$$

G: abelian group

$$\Delta^{\mathcal{U}}_*(X) \otimes G \to \Delta_*(X) \otimes G$$

chains with coefficients in G (form: $\sum g_i \sigma_i$)

induce isomorphisms in cohomology. $\square$

Let $X = \overset{\circ}{A} \cup \overset{\circ}{B}$ (interior) $\quad \mathcal{U} = \{A, B\}$

One has the exact sequence ; $j_A : A \hookrightarrow A \cup B$

$i_A : A \cap B \hookrightarrow A$

$$\boxed{0 \to \Delta_p(A \cap B) \xrightarrow{i_A \oplus i_B} \Delta_p(A) \oplus \Delta_p(B) \xrightarrow{j_A - j_B} \Delta_p^{\mathcal{U}}(A \cup B) \to 0}$$

(free abelian groups)

attention

Applying $\mathrm{Hom}\,(\cdot, G)$ or $\otimes G$ yields a short exact sequence which, by Mayer-Vietoris, gives

this is taken for granted (it is not complicated)

$$\cdots \to H^p(A \cup B) \xrightarrow{j_A^* - j_B^*} H^p(A) \oplus H^p(B) \xrightarrow{i_A^* + i_B^*} H^p(A \cap B) \to$$
$$\to H^{p+1}(A \cup B) \to \cdots \quad \forall \text{ coefficient group } G.$$

The de Rham theorem

Remark: we shall take M oriented however, it would be necessary to orient Δ_p only

Recall

$$\sigma : \Delta_p \to M$$

smooth singular simplex

$$(\sigma : \Omega \supset \Delta_p \to M)$$

"plane"

wlog since $H^{\infty}_{sing} = H_{sing}$

orientation for Δ_p :

one chooses the positive one for Δ_0, then given the orientation for Δ_{p-1}, one orients $\partial\Delta_p$ in such a way that the face map

$$F_0 : \Delta_{p-1} \to \partial\Delta_p$$

preserves orientation.

Finally one orients Δ_p consistently with $\partial\Delta_p$

Then set

$$\int_\sigma \omega := \int_{\Delta_p} \sigma^* \omega$$

and, for $c = \sum n_i \sigma_i$

$$\left[\int_c \omega = \sum n_i \int_{\sigma_i} \omega \right]$$

The following homomorfism is defined

$$\boxed{\begin{aligned} \psi(\omega) &: \Delta_p(M) \to \mathbb{R} \\ \psi(\omega)(c) &= \int_c \omega \end{aligned}}$$

$\Rightarrow$

$$\psi : \Omega^p(M) \longrightarrow \underbrace{\mathrm{Hom}(\Delta_p(M), \mathbb{R})}_{\Delta^p(M,\mathbb{R})}$$

homomorphisms

$$= \Delta_p(M,\mathbb{R})^*$$

$\Delta^p(M,\mathbb{R})$ smooth singular p-cochains

let

$$\delta : \Delta^{p-1}(M,\mathbb{R}) \to \Delta^p(M,\mathbb{R})$$

$$(\delta f)(c) := f(\partial c)$$

δ = transpose of ∂

$\Rightarrow$ one gets $(\Delta^*(M,\mathbb{R}), \delta)$

complex consisting all smooth singular p-cochains

$$\boxed{H^*(M,\mathbb{R}) := H^*(\Delta^*(M,\mathbb{R}))}$$

smooth singular cohomology

the diagram

$$\begin{array}{ccc} \Omega^{p-1}(M) & \xrightarrow{\Psi} & \Delta^{p-1}_{\cdot}(M, \mathbb{R}) \\ \downarrow d & & \downarrow \delta \quad (= \mathrm{Hom}(\partial, 1)) \\ \Omega^{p}(M) & \xrightarrow{\Psi} & \Delta^{p}_{\cdot}(M, \mathbb{R}) \end{array}$$

is commutative, namely

⚡ Ψ is a chain map

Proof: ⚡ Stokes :

$$\boxed{\Psi(d\omega)(\sigma)} = \int_\sigma d\omega = \int_{\Delta_p} \sigma^*(d\omega) =$$

$$= \int_{\Delta_p} d(\sigma^*\omega) = \int_{\partial\Delta_p} \sigma^*\omega =$$

$$= \sum_i (-1)^i \int_{\Delta_{p-1}} F_i^* \sigma^* \omega = \sum (-1)^i \int_{\Delta_{p-1}} (\sigma \circ F_i)^* \omega$$

$$= \sum_i (-1)^i \int_{\sigma \circ F_i} \omega = \int_{\partial\sigma} \omega = \Psi(\omega)(\partial\sigma)$$

$$\equiv \underline{\delta(\Psi(\omega))(\sigma)}$$

One has then an induced homomorphism in cohomology

$\int_{c+\partial b} \omega + da = \ldots$

$\int_c \omega$ if $d\omega = \partial c = 0$

$$\psi^* \quad H^p_{DR}(M) \longrightarrow H^p(M, \mathbb{R})$$

ψ^* is well-defined.

$$[\omega] \longmapsto (c \mapsto \psi(\omega)(c) = \int_c \omega)$$

Theorem of de Rham : ψ^* is an isomorphism

Proof (we shall confine ourselves to manifolds having a finite good cover (compact manifolds fall into this class)

The diagram below is commutative and with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \Omega^p(U \cup V) & \to & \Omega^p(U) \oplus \Omega^p(V) & \to & \Omega^p(U \cap V) & \to & 0 \\ & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \\ 0 & \to & \Delta^p_{\mathcal{U}}(U \cup V) & \to & \Delta^p(U) \oplus \Delta^p(V) & \to & \Delta^p(U \cap V) & \to & 0 \end{array}$$

⚠ "Small" cochains

$\Downarrow$ Mayer-Vietoris

$$\begin{array}{ccccccccc} \cdots \to & H^p_{DR}(U \cup V) & \to & H^p_{DR}(U) \oplus H^p_{DR}(V) & \to & H^p_{DR}(U \cap V) & \xrightarrow{d^*} & H^{p+1}_{DR}(U \cup V) \\ & \downarrow \psi^* & & \downarrow \psi^* & & \downarrow \psi^* & & \downarrow \psi^* \\ \cdots \to & H^p(U \cup V) & \to & H^p(U) \oplus H^p(V) & \to & H^p(U \cap V) & \to & H^{p+1}(U \cup V) \end{array}$$

☆ From the five lemma and from Mayer Vietoris it follows that if ψ^* is an isomorphism for $U, V, U \cap V$, it is such for $U \cup V$

☆ From the Poincaré lemmata and the other assumptions the thesis follows ("Mayer-Vietoris")

+ cone construction (⚠ the latter does not produce smooth simplices but the result is unchanged) ; □

Remark

In Singer-Thorpe de Rham is proven for simplicial cohomology and for differentiably triangulated manifolds. Compact manifolds are such. In our case (see Bredon) the proof holds for all manifolds. but, for simplicity, we confined ourselves to those possessing a finite good covering.

Some extra details

The position $c \mapsto \int_c \omega = \psi(\omega)(c)$ defines $\psi(\omega) \in \mathrm{Hom}(\Delta_p(M), \mathbb{R})$

$$\mathrm{Hom}(\Delta_p(M), \mathbb{R}) \cong \Delta^p(M, \mathbb{R})$$

Observe that

$$\psi(d\omega) = \int_c d\omega = \int_{\partial c} \omega = \psi(\omega)(\partial c)$$

$$= \delta[\psi(\omega)](c)$$ i.e. ; $d\omega \mapsto$ coboundary

Moreover, on hvs $\left(\int_{c+\partial b} \omega + da = \ldots \int_c \omega \right.$

if $\partial c = 0$, $\left. d\omega = 0 \right)$

Mo

$$(**)\quad \boxed{\begin{tikzcd} H^*_{DR}(M) \arrow[r, "\Psi^*", "\cong"'] \arrow[rr, bend right, "\Phi^*"'] & H^*(M, \mathbb{R}) \arrow[r, "\tau", "\cong"'] & \left[H_*(M, \mathbb{R})\right]^* \end{tikzcd}}$$

con $\Phi^*: [\omega] \longrightarrow \left([c] \to \Psi(\omega)(c) \right)$

$$\Psi(\omega)(c) = \int_c \omega$$

★ Description of τ

$f : c \to f(c)$

(c) f $\quad \delta f = 0 \quad (f(\partial b) = 0 \quad \forall b)$

$\tau : f \longmapsto (f : [c] \mapsto f(c))$

or better $[f]$ $\qquad f(\partial b) = \delta f(b) = 0$

$\delta g \mapsto \delta g(c) = g(\partial c) = 0$

$\partial c = 0$

Conversely, given $f \in H_*(M, \mathbb{R})^*$:

$f(c + \partial b) = f(c)$ & $\partial c = 0$, it can be arbitrarily extended to

★

$\Delta_* = Z_p \oplus A$

$A \cong \Delta_* / Z_p$ $\qquad \delta f(c) = f(\partial c) = f(0) = 0$

★ Consequences

1) Let $w \in Z^p(M)$ such that

$$\int_c w = 0 \quad \forall \quad c \in Z_p(M)$$

(integral coefficients)

Allora w is exact ★

Proof. One has

$$\psi^*([w]) = 0$$

$(\psi^*([w]) \in H^p(M, \mathbb{R})$
$\psi^*([w])([c]) = \int_c w$

$\Rightarrow$ (ψ^* is an isomorphism) ★ $w = 0$ in $H^p(M)$

i.e. $w = da$.

⚠

2) If for fixed $c \in H_p(M)$, one has

$$\int_c w = 0 \quad \forall \quad w \in Z^p \quad \text{then}$$

$$\psi^*(w)(c) = 0 \quad \forall \quad [w] \in H^p$$

Therefore $[c]$ defines an element in $H_{DR}(M)^*$, the zero element.

Now $H^*_{DR}(M)^* \cong H^*_{sing}(M)^*$

which coincides with $H_{*sing}(M, \mathbb{R})$ if its dim. is finite

for instance, if M has a finite good covering

★ ★
$\Rightarrow [c]$ is a priori torsion (In case there is no torsion, $[c] = 0$

see also below

i.e. $k[c] = 0$ for $k \neq 0 \Rightarrow c = \partial b$)