Lectures on
DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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Lecture XL

MANIFOLDS WITH BOUNDARY. STOKES' THEOREM

✱ Manifolds with boundary

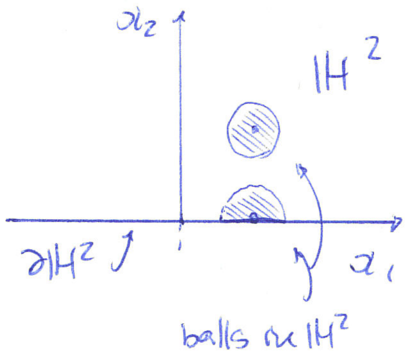
A smooth manifold with boundary, of dimension n is a Hausdorff second countable topological space equipped with an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$, with $U_\alpha \simeq$ ball in $\mathbb{R}^n$ (homeomorphic)

or $U_\alpha \simeq$ ball in $\mathbb{H}^n = \{(x_1 \ldots x_n) \in \mathbb{R}^n \,/\, x_n \geq 0\}$

upper half-space (equipped with the relative topology inherited from $\mathbb{R}^n$)

(with smooth transition maps, of course)

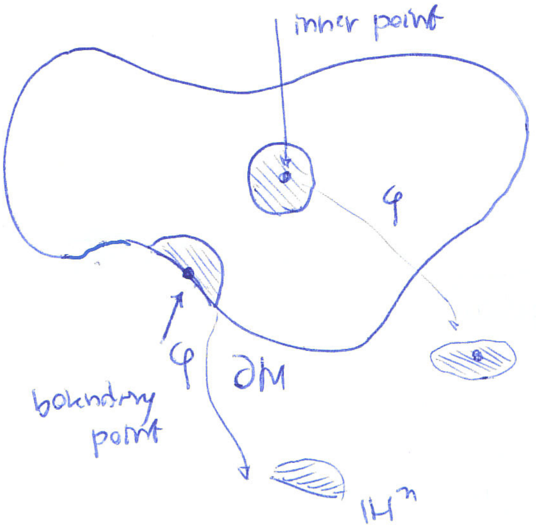
The boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$ is: $\boxed{\partial\mathbb{H}^n = \{(x_1 \ldots x_n) \,/\, x_n = 0\}} \cong \mathbb{R}^{n-1}$



The boundary ∂M of M is then defined as follows:

$$\boxed{\partial M := \{p \in M \,/\, \exists \text{ local chart } \varphi \text{ with } \varphi(p) \in \partial\mathbb{H}^n\}}$$

The boundary ∂M is a $(n-1)$-dimensional manifold



Aside

Beware of the difference between topological boundaries and manifold boundaries (in the sequel we shall always deal with manifold boundaries)

- $\overline{D^2}$ closed unit disc in $\mathbb{R}^2$ (with relative topology)

topological boundary $\partial_{top} X = \overline{X} \setminus \overset{\circ}{X}$

$\uparrow$ closure $\quad \uparrow$ interior

$\partial_{top} \overline{D^2} = \overline{D^2} \setminus D^2 = S^1$

$\partial \overline{D^2} = S^1$

$\partial \overline{D^2}$

$\uparrow$ manifold sense

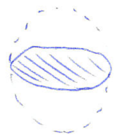
$\Rightarrow$ in this case they agree.

- $X = \overline{D}^2$ as a topological space in itself.

one has $X = \overline{X} = \overset{\circ}{X} \Rightarrow \partial_{top} \overline{D}^2 = \emptyset$!

- $X = \overline{D}^2 \subset \mathbb{R}^3$

as a subset of $\mathbb{R}^3$



$\overline{X} = X \qquad \overset{\circ}{X} = \emptyset \qquad \partial_{top} X = X$!

- $\partial D^2 = \emptyset$ (but notice that a boundary can be adjoined to any ordinary manifold (via $x_n \to y = e^{x_n} > 0$.))

$\partial_{top} D^2 = \overline{D}^2 \setminus D^2 = S^1$ (as a subset of $\mathbb{R}^2$)

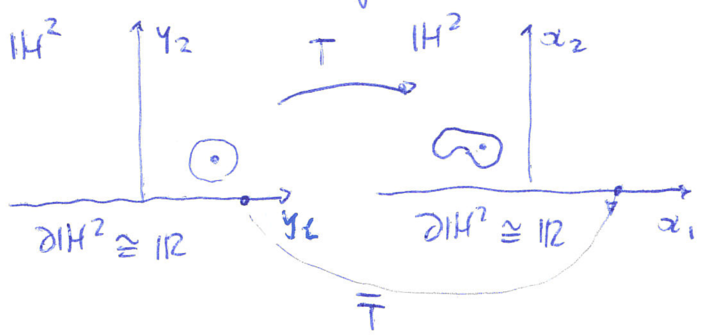
in itself : $\partial_{top} D^2 = \emptyset$

☆ We shall now show that, given an orientation on M, (it is M orientable, of course) then it induces a natural orientation on ∂M (symbolically: $[M] \to [\partial M]$)

This is implied by the following general fact (which we shall check just for $n=2$)

☆ Let $T : \mathbb{H}^n \to \mathbb{H}^n$ be a diffeomorphism with $J(T) > 0$ (Jacobian)
(assume it is defined on an open set in $\mathbb{R}^n$ containing $\mathbb{H}^n$...). Then T induces a diffeomorphism $\bar{T} : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ (it is a diffeomorphism of $\mathbb{R}^{n-1}$) with $J(\bar{T}) > 0$

Dim ($n=2$) Clearly, $\bar{T}$ is well-defined, since interior points are mapped to interior points, in view of the inverse function theorem (therefore, boundary points are mapped to boundary points)



Let

$$T : \begin{cases} x_1 = T_1(y_1, y_2) \\ x_2 = T_2(y_1, y_2) \end{cases}$$

$$\bar{T} : \{ x_1 = T_1(y_1, 0) \}$$

(Notice that $T_2(y_1, 0) = 0 \quad \forall\, y_1 \in \mathbb{R}$, in view of the previous remarks). Let us compute

$$J(T)(y_1, 0) = \begin{vmatrix} \frac{\partial T_1}{\partial y_1}(y_1, 0) & \frac{\partial T_1}{\partial y_2}(y_1, 0) \\ \frac{\partial T_2}{\partial y_1}(y_1, 0) & \frac{\partial T_2}{\partial y_2}(y_1, 0) \end{vmatrix} > 0$$

$$\Rightarrow \frac{\partial T_2}{\partial y_1}(y_1,0) = 0$$

hence

$$J(T)(y_1,0) = \underbrace{\frac{\partial T_1}{\partial y_1}(y_1,0)}_{J(\overline{T})(y_1)} \, \frac{\partial T_2}{\partial y_2}(y_1,0) > 0$$

The conclusion will then be achieved once we show that

$$\frac{\partial T_2}{\partial y_2}(y_1,0) > 0$$

Consider

$$\frac{T_2(y_1,y_2) - \overbrace{T_2(y_1,0)}^{=0}}{y_2} = \frac{\overbrace{T_2(y_1,y_2)}^{>0}}{\underbrace{y_2}_{>0}}$$

Taking $y_2 \to 0$, we have

$$\frac{\partial T_2}{\partial y_2}(y_1,0) \geq 0$$

but $=$ is excluded $(J(T) > 0)$,

whence $J(\overline{T}) > 0$.

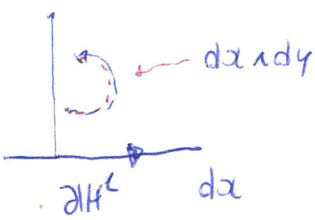
Now set $[\mathbb{H}^n] = dx_1 \wedge \dots \wedge dx_n$ standard orientation of $\mathbb{H}^n$

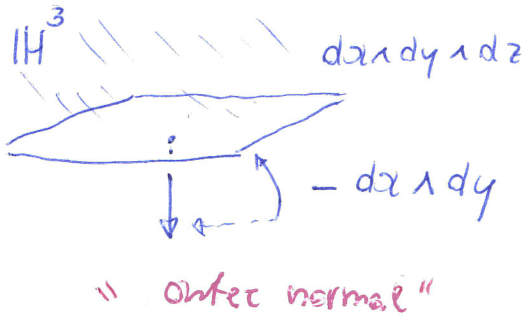
The <u>induced</u> orientation will be

$$[\partial \mathbb{H}^n] = (-1)^n dx_1 \wedge dx_2 \wedge \dots \wedge dx_{n-1}$$

Examples:

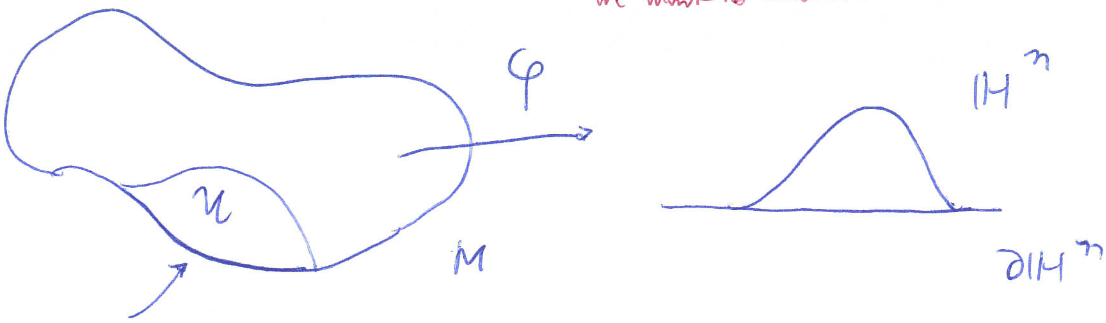
$\mathbb{H}^2$





So, given $[M]$, $[\partial M]$ is constructed as follows

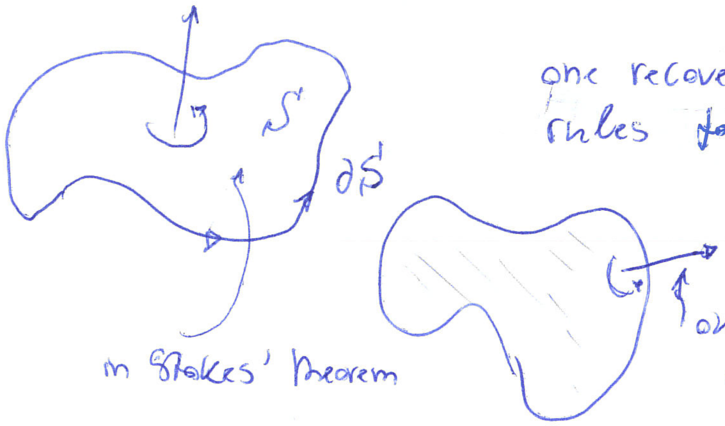
we want to construct never vanishing forms...



$\partial\mathcal{U} := \partial M \cap \mathcal{U}$

$$[\partial M]\Big|_{\partial\mathcal{U}} := \varphi^*[\partial\mathbb{H}^n]$$

example:



in Stokes' theorem

one recovers the ordinary rules ~~for~~ orienting boundaries in standard vector analysis

outer normal

(in the divergence theorem)

We are now prepared to state the basic

Stokes' theorem

Let $\omega \in \Lambda_c^{n-1}(M)$ * compactly supported, with M oriented, ∂M with the induced orientation

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

Proof. It will be achieved in three steps.

(1) The theorem is true for $\mathbb{R}^n$

Let $\omega = f(x_1 \dots x_n)\, dx_1 \wedge \dots dx_{n-1}$ (This is enough; one extends by linearity)

f compactly supported, smooth

$$d\omega = \pm \frac{\partial f}{\partial x_n} dx_1 \wedge \dots dx_{n-1} \wedge dx_n$$

$$\boxed{\int_a^b \xi' \, dt = \xi(b) - \xi(a)}$$

Then $$\int_{\mathbb{R}^n} d\omega \underset{(\text{Fubini})}{=} \pm \int_{\mathbb{R}^{n-1}} dx_1 \wedge \dots dx_{n-1} \underbrace{\int_{-\infty}^{+\infty} \frac{\partial f}{\partial x_n} dx_n}_{= 0}$$

$$\Rightarrow \int_{\mathbb{R}^n} d\omega = 0 \qquad (f \text{ has compact support})$$

But $\partial \mathbb{R}^n = \emptyset$, therefore $\int_{\partial \mathbb{R}^n} \omega = 0$ ✓

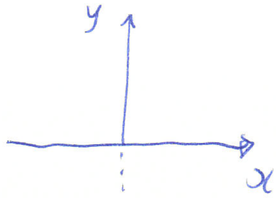
(2) The theorem holds for $\mathbb{H}^n$.

Let us check $n = 2$ (without substantial loss of generality)

$\omega = f(x,y)\, dx + g(x,y)\, dy$ $\qquad f, g$ smooth

We find, successively, from

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$



$$\int_{\mathbb{H}^2} d\omega = -\int_{\mathbb{H}^2} \frac{\partial f}{\partial y}\, dx \wedge dy + \int_{\mathbb{H}^2} \frac{\partial g}{\partial x}\, dx \wedge dy$$

$$= -\int_{-\infty}^{+\infty} \Big(\underbrace{\int_0^{+\infty} \frac{\partial f}{\partial y}\, dy}_{f(x,\infty) - f(x,0)} \Big)\, dx + \int_0^{+\infty} \Big(\underbrace{\int_{-\infty}^{+\infty} \frac{\partial g}{\partial x}\, dx}_{=\,0} \Big)\, dy$$

$f(x,\infty) = 0$ ← abuse of notation (f is compactly supported)

0 (g has compact support)

$$= + \int_{-\infty}^{+\infty} f(x,0)\, dx = \int_{\partial \mathbb{H}^2 = \mathbb{R}} \omega \qquad \checkmark$$

(3) We collect the previous findings: let

$$\omega = \sum_{\alpha \in \Omega} \rho_\alpha \omega \qquad \left(\text{partition of unity, recall } \sum_\alpha \rho_\alpha \equiv 1\right)$$

← this is always a finite sum

In view of linearity, it is enough to work with a single form $\rho_\alpha \omega$, whose (compact) support is in $\mathcal{U}_\alpha$. But, since $\mathcal{U}_\alpha$ is diffeomorphic either to a ball in $\mathbb{R}^n$ or to a ball in $\mathbb{H}^n$, we may use steps (1) and (2):

$$\int_M d(\rho_\alpha \omega) = \int_{\mathcal{U}_\alpha} d(\rho_\alpha \omega) \underset{\substack{\uparrow \\ \text{Stokes for } \mathbb{R}^n \\ \text{and } \mathbb{H}^n}}{=} \int_{\partial \mathcal{U}_\alpha} \rho_\alpha \omega = \int_{\partial M} \rho_\alpha \omega$$

and this concludes the proof. $\square$