Lezione XIV

MATEMATICHE COMPLEMENTARI II

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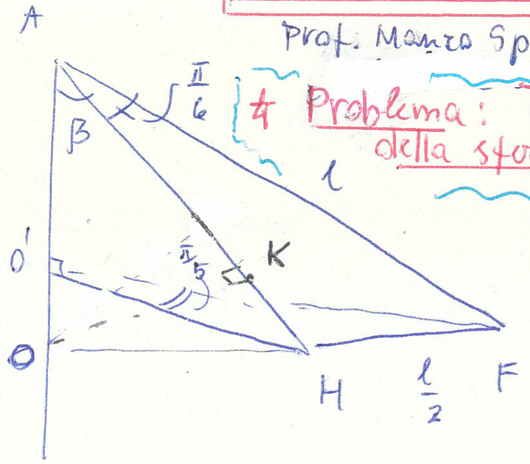
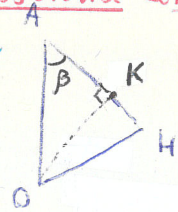
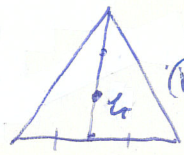
* Complementi alla teoria dei solidi platonici

* Problema: Trovare AO, raggio della sfera circoscritta all'icosaedro di lato ℓ





* K è il baricentro di ABF $\Rightarrow$ $AK = \frac{2}{3} AH$ (vedi oltre)



$$AO = \frac{AK}{\cos\beta} = \frac{2}{3} AH \frac{1}{\cos\beta} \qquad \frac{2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}$$

Ma $AH = \frac{\sqrt{3}}{2}\ell \Rightarrow$

$$\boxed{AO = \frac{1}{\sqrt{3}} \ell \frac{1}{\cos\beta} \qquad (*)}$$

ora è pure

$$AH = O'H \frac{1}{\sin\beta}$$

e

$$\frac{HF}{O'H} = \tan\frac{\pi}{5} \Rightarrow O'H = HF \operatorname{cotan}\frac{\pi}{5} = \frac{\ell}{2}\operatorname{cotan}\frac{\pi}{5}$$

$$\Rightarrow AH = \frac{\cancel{\ell}}{\cancel{2}} \operatorname{cotan}\frac{\pi}{5} \cdot \frac{1}{\sin\beta} = \frac{\sqrt{3}}{\cancel{2}}\cancel{\ell}$$

$$\Rightarrow \sin\beta = \frac{1}{\sqrt{3}} \operatorname{cotan}\frac{\pi}{5}$$

$\Rightarrow$ si ricava $\cos\beta$ e quindi AO dalla (*)

$$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4} = \frac{\phi}{2} \qquad \phi = \frac{1+\sqrt{5}}{2}$$

$$\sin\frac{\pi}{5} = \sqrt{1-\frac{\phi^2}{4}} = \frac{\sqrt{4-\phi^2}}{2}$$

$$\cot\frac{\pi}{5} = \frac{\cos}{\sin} = \frac{\phi}{\sqrt{4-\phi^2}} \qquad \sin\beta = \frac{1}{\sqrt{3}}\ \frac{\phi}{\sqrt{4-\phi^2}}$$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{\phi^2}{3(4-\phi^2)}} =$$

$$= \sqrt{\frac{3(4-\phi^2)-\phi^2}{3(4-\phi^2)}} = \sqrt{\frac{12-3\phi^2-\phi^2}{3(4-\phi^2)}} =$$

$$= \sqrt{\frac{12-4\phi^2}{3(4-\phi^2)}} = \sqrt{\frac{12-4\phi^2}{12-3\phi^2}}$$

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \phi^2 = \frac{1+5+2\sqrt{5}}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

$$12-4\phi^2 = 12-\cancel{4}^{2}\,\frac{3+\sqrt{5}}{\cancel{2}} = 12-2(3+\sqrt{5}) = 12-6-2\sqrt{5}$$

$$= 6-2\sqrt{5}$$

$$= 2(3-\sqrt{5})$$

$$12-3\phi^2 = 12-4\phi^2+\phi^2 =$$

$$= 6-2\sqrt{5}+\frac{3+\sqrt{5}}{2} = \frac{12-4\sqrt{5}+3+\sqrt{5}}{2} = \frac{15-3\sqrt{5}}{2}$$

$$= \frac{3}{2}(5-\sqrt{5})$$

$$\cos\beta = \sqrt{\frac{2(3-\sqrt{5})}{\frac{3}{2}(5-\sqrt{5})}} = \sqrt{\frac{4(3-\sqrt{5})}{3(5-\sqrt{5})}}$$

$$\frac{1}{\cos\beta} = \sqrt{\frac{3(5-\sqrt{5})}{4(3-\sqrt{5})}} = \frac{\sqrt{3}}{2}\sqrt{\frac{5-\sqrt{5}}{3-\sqrt{5}}}$$

$$= \frac{\sqrt{3}}{2}\cdot\sqrt{\frac{(5-\sqrt{5})(3+\sqrt{5})}{9-5}} = \frac{\sqrt{3}}{2}\sqrt{\frac{15-3\sqrt{5}+5\sqrt{5}-5}{4}}$$

$$= \frac{\sqrt{3}}{2}\sqrt{\frac{10+2\sqrt{5}}{4}} = \frac{\sqrt{3}}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$$

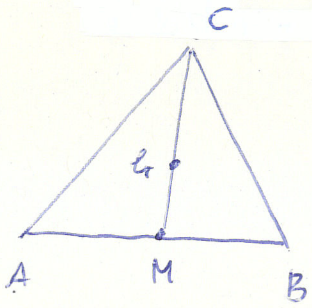
4

$$AO = \frac{1}{\sqrt{3}}\, l \cdot \underbrace{\frac{\sqrt{3}}{2}\cdot\sqrt{\frac{5+\sqrt{5}}{2}}}_{\frac{1}{\cos\beta}} = \frac{l}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$$

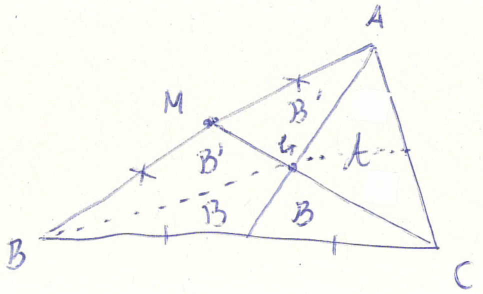
RICHIAMO !

Le mediane di un triangolo si incontrano in un pto (baricentro), e si ha, ad es.

$$\overline{CG} = 2\,\overline{GM}$$



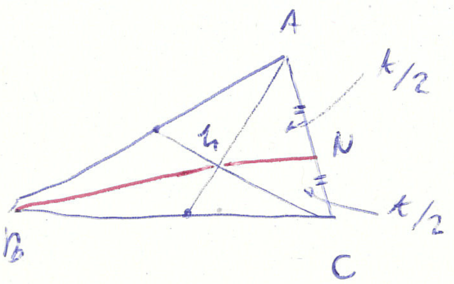
Dim. Consideriamo due mediane, si chiami G la loro intersezione. Applicando ripetutamente i teoremi del I° libro si ha (v. figura accanto)

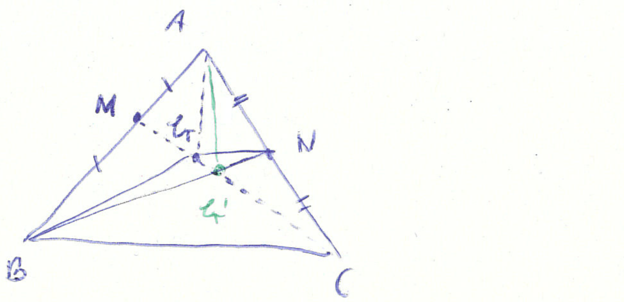


$$A + B = B + 2B'$$

$$\Rightarrow \quad A = 2B' \quad \Rightarrow \quad \overline{MG} = \frac{1}{2}\,\overline{CG}$$

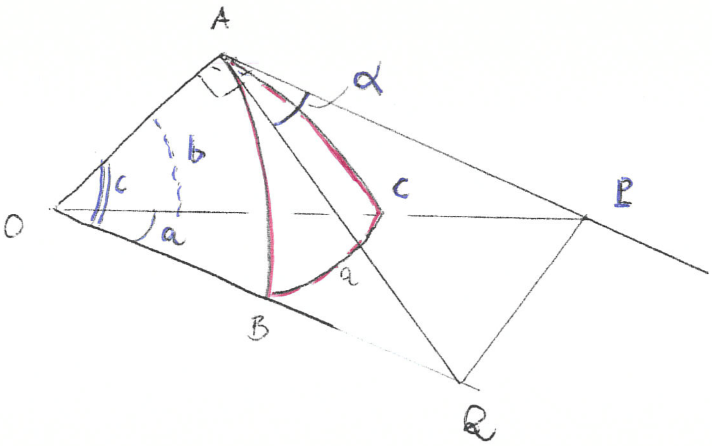
Facciamo ora vedere che B, G, N sono allineati

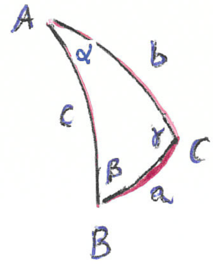




Se $G \neq G'$ si ha $MG = \frac{1}{2} GC$ e $MG' = \frac{1}{2} G'C$, assurdo. Dunque $G = G'$.

4 "Formula di Carnot" per i triangoli sferici





$$\overline{PQ}^2 = \overline{PO}^2 + \overline{QO}^2 - 2\,\overline{PO}\cdot\overline{QO}\,\cos a$$
$$\overline{PQ}^2 = \overline{PA}^2 + \overline{QA}^2 - 2\,\overline{PA}\cdot\overline{QA}\,\cos \alpha$$

$$\underbrace{\overline{PO}^2 - \overline{PA}^2}_{=\,\overline{OA}^2} + \underbrace{\overline{QO}^2 - \overline{QA}^2}_{\overline{OA}^2} - 2\,\overline{PO}\cdot\overline{QO}\,\cos a + 2\,\overline{PA}\cdot\overline{QA}\,\cos\alpha = 0$$

$$\overline{OA}^2 + \overline{PA}\cdot\overline{QA}\,\cos\alpha = \overline{PO}\cdot\overline{QO}\,\cos a$$

$$\frac{\overline{OA}\cdot\overline{OA}}{\overline{OP}\cdot\overline{OQ}} + \frac{\overline{PA}}{\overline{PO}}\,\frac{\overline{QA}}{\overline{QO}}\,\cos\alpha = \cos a$$

$$\boxed{\cos a = \cos b \cos c + \sin b \sin c \cos\alpha}$$

Per triangoli "piccoli" $\cos a \sim 1 - \frac{a^2}{2}$ $\quad \sin a \sim a$

(fino all'ordine 2)

$$1 - \frac{a^2}{2} = \left(1 - \frac{b^2}{2}\right)\left(1 - \frac{c^2}{2}\right) + bc\cos\alpha + \dots$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos\alpha + \dots$$

← CARNOT piano

4 Teorema dei seni

$$\cos b \cos c - \cos a = -\sin b \sin c \cos\alpha$$

quadrando, si ha:

$$\underbrace{\cos^2 b}_{1-\sin^2 b}\ \underbrace{\cos^2 c}_{1-\sin^2 c} + \underbrace{\cos^2 a}_{1-\sin^2 a} - 2\cos a\cos b\cos c = \sin^2 b \sin^2 c \cos^2\alpha$$

$$1 - \sin^2 b - \sin^2 c + \sin^2 b \sin^2 c + 1 - \sin^2 a - 2\cos a \cos b \cos c$$

$$= \sin^2 b \sin^2 c \times (1 - \sin^2\alpha)$$

è simmetrica rispetto ad a, b, c

$$\underbrace{2 - \sin^2 b - \sin^2 c - \sin^2 a - 2\cos a\cos b\cos c}$$

$$= -\sin^2 b \sin^2 c \sin^2\alpha$$

$$\Rightarrow \qquad \sin^2 b \sin^2 c \sin^2\alpha = \sin^2 b \sin^2 a \sin^2\gamma$$

ad es.

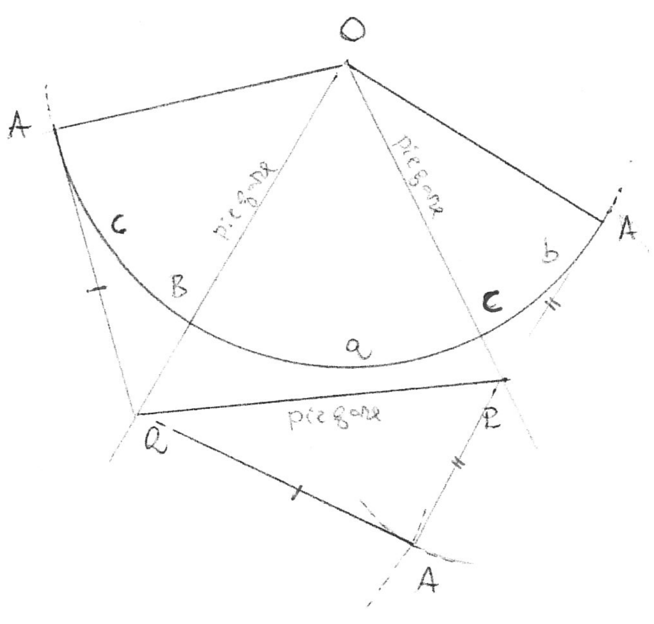
$0 < \sin(\cdot) \le 1$

per tutti gli angoli

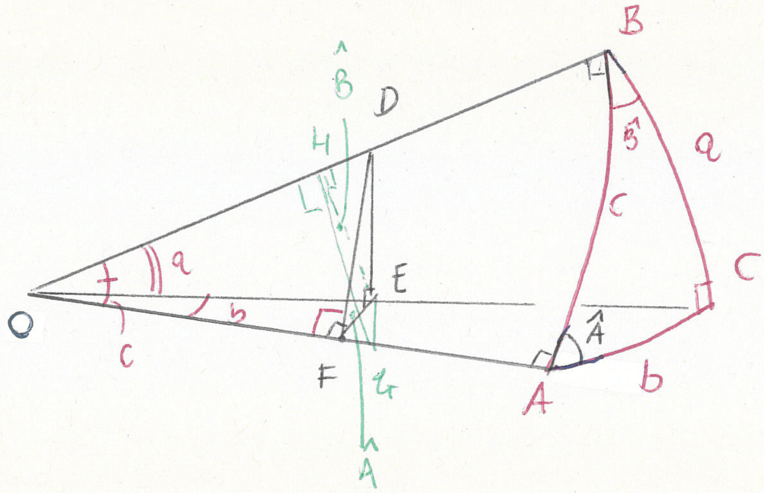
$$\boxed{\frac{\sin a}{\sin\alpha} = \frac{\sin c}{\sin\gamma} \left(= \frac{\sin b}{\sin\beta} \right)}$$

per triangoli "piccoli" $\sin a \sim a \Rightarrow$ si ha il teorema dei seni piano.

come costruire un triangolo sferico



Formule di Napier per triangoli rettangoli sferici



Si osservi che $O\hat{D}F = \frac{\pi}{2}$

infatti $OD^2 \underset{I.47}{=} OE^2 + DE^2 \underset{I.47}{=} OF^2 + FE^2 + DE^2 \underset{I.47}{=} OF^2 + DF^2$

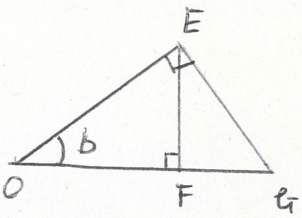
$\Rightarrow$ (I.48), $O\hat{D}F$ è retto.

angolo tra i piani $\pi(OAB)=\pi(OFD)$ e $\pi(OAC)=\pi(OFE)$

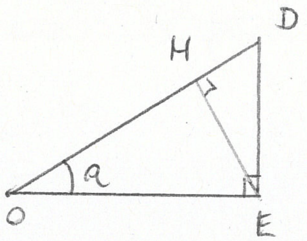
ora $\sin a = \frac{DE}{OD} = \frac{DE}{DF} \cdot \frac{DF}{OD} = \sin\hat{A}\,\sin c$ segue anche subito dal teorema dei seni

altro esempio! $\boxed{\sin b = \frac{EF}{OE} = \frac{EF}{DE} \cdot \frac{DE}{OE} = \cot\hat{A}\,\tan a}$

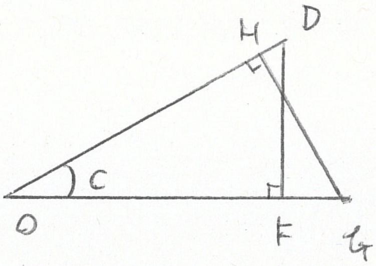
ecc. Si costruisca poi (v. figura) E G H (si formano vari triangoli)



sul piano OAC



sul piano OCB



sul piano OAB

Calcoliamo:

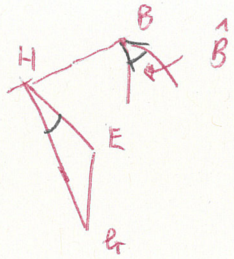
$$\cos c = \frac{OF}{OD} = \frac{OF}{OE}\cdot\frac{OE}{OD} = \cos b \cdot \cos a \qquad (\text{... già nota})$$

(simil.) $\Vert$

$$\frac{EF}{EG}\cdot\frac{EH}{DE} = \frac{EF}{DE}\cdot\frac{EH}{EG} = \cot\hat{A}\cdot\cot\hat{B}$$

$$\boxed{\cos c = \cot\hat{A}\ \cot\hat{B}}$$

!



Analogamente:

$$\cos\hat{A} = \frac{EF}{DF}$$

$$\cos a = \frac{OE}{OD} = \frac{OH}{OE}$$

$$\sin\hat{B} = \frac{EG}{HG}$$

$$\cos a \,\sin\hat{B} = \frac{OH}{OE}\,\frac{EG}{HG} = \underset{\underset{\cot c\ \tan b}{\Vert}}{\frac{OH}{HG}}\,\frac{EG}{OE} = \frac{OF}{DF}\cdot\frac{EF}{OF} = \frac{EF}{DF} = \cos\hat{A}$$

$$\boxed{\cos\hat{A} = \sin\hat{B}\,\cos a = \cot c\ \tan b}$$

* ve ne sono molte altre, codificabili tramite il "pentagramma mirificum"

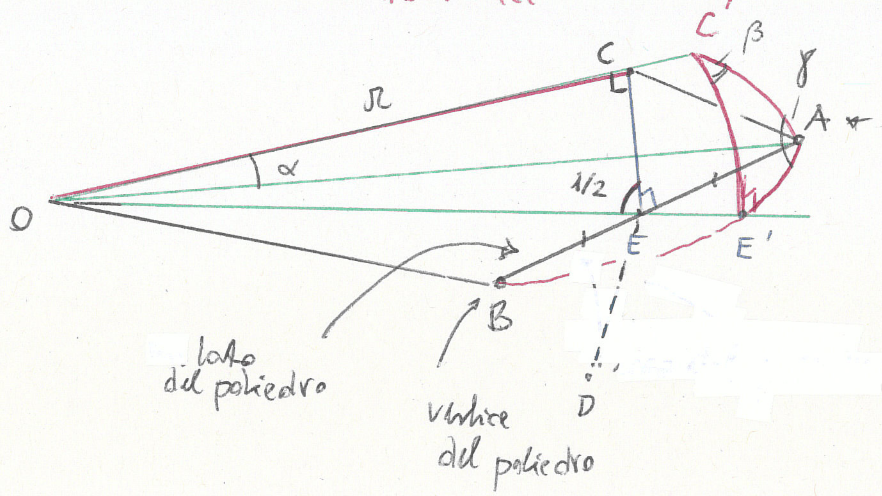
✦ raggio della sfera inscritta in un poliedro regolare
(a sua volta inscritto in una sfera unitaria)

m : lati di una singola faccia

n : # facce comuni ad un dato vertice

$\overline{OA} = \overline{OB} = \overline{OC'} = 1$

$\overline{OC} = r = ?$

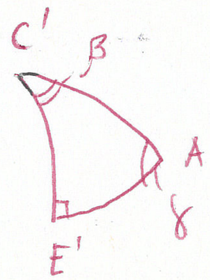


	m	n
tetraedro:	3	3
cubo	4	3
ottaedro	3	4
dodecaedro	5	3
icosaedro	3	5

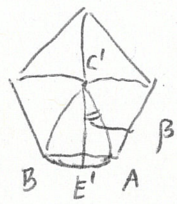
$$r = \cos \widehat{AC'} = \cos\alpha$$

Ma, per una delle formule di Napier, è

$$\underbrace{\cos \widehat{AC'}}_{\alpha} = \cot \underbrace{A\hat{C}'E'}_{\beta} \cot \underbrace{C'\hat{A}E'}_{\gamma}$$

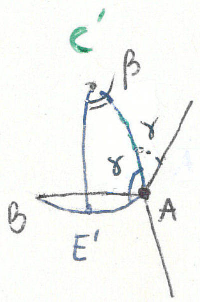


Ora $\beta = A\hat{C}'E' = \dfrac{2\pi}{2m} = \dfrac{\pi}{m}$



es: →

m lati di una faccia



$$\gamma = \frac{2\pi}{2n} = \frac{\pi}{n}$$

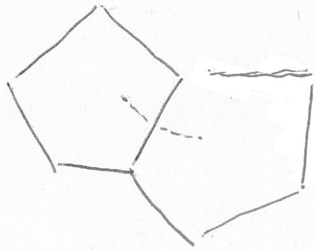
n = # facce che insistono su un vertice

In definitiva:

$$\boxed{r = \cot\frac{\pi}{m}\cot\frac{\pi}{n}}$$

✦ solidi duali hanno lo stesso r

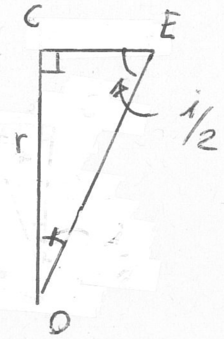
Inclinazione di due facce



con riferimento alla figura

e ancora per Napier

$$\cos \underbrace{C'\hat{A}E'}_{\gamma} = \sin \underbrace{A\hat{C'}E'}_{\beta} \cdot \cos \widehat{C'E'}$$



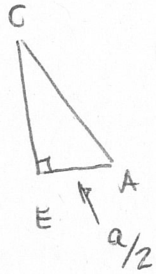
$$= \sin\beta \cdot \cos C\hat{O}E$$

$$= \sin\beta \, \sin\frac{i}{2}$$

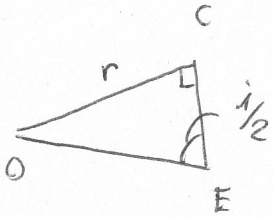
$$\Rightarrow \boxed{\sin\frac{i}{2} = \frac{\cos(\pi/n)}{\sin(\pi/m)}}$$

Calcolo di a = lato del poliedro inscritto nella sfera unitaria

$$\frac{CE}{AE} = \tan C\hat{A}E = \cot A\hat{C}E = \cot\frac{\pi}{m}$$



$$\overline{AE} = \frac{a}{2} \quad \Rightarrow \quad \overline{CE} = \frac{a}{2}\cot\frac{\pi}{m}$$



ma $r = CE \tan\frac{i}{2}$

$$\Rightarrow CE = r\cot\frac{i}{2} = \cot\frac{\pi}{m}\cot\frac{\pi}{n}\cot\frac{i}{2}$$

$$\Rightarrow \boxed{a = 2\cdot\cot\frac{\pi}{n}\cot\frac{i}{2}}$$