FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
DIPARTIMENTO DI MATEMATICA E FISICA "NICCOLÒ TARTAGLIA"

Topics in symplectic and multisymplectic geometry

International Ph.D. Programme in "Science"
Project: Differential geometry and applications to modern physics
Coordinators: Mauro Spera (UCSC) and Marco Zambon (KU Leuven)

Prof. MAURO SPERA
Università Cattolica del Sacro Cuore

The course is intended as an introduction to the methods of symplectic and multisymplectic geometry, with a view to their multifaceted physical applications.
Here is a cursory and tentative list of the planned topics: Symplectic manifolds, moment maps and reduction, geometric fluid mechanics, covariant phase space, multisymplectic manifolds, conserved quantities, geometric quantization of line bundles and gerbes.

Basic acquaintance with differential geometry is required; however, specific technical tools will be developed when needed.
Handwritten lecture notes will be gradually made available online.

Ph.D. Course

Gennaio-Aprile 2017
Inizio: martedì 10 gennaio 2017
Aula 7, ore 14.30 - 16.30
Via dei Musei 41 - Brescia

UNIVERSITÀ
CATTOLICA
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January 2017: Tue. 10, Wed. 11, Tue. 17, Wed. 18, Wed. 25, Tue. 31
February 2017: Wed. 1, Tue. 7, Tue. 14, Tue. 21, Wed. 22, Tue. 28
March 2017: Wed. 1, Tue. 7, Wed. 8, Tue. 14, Wed. 15, Tue. 21, Wed. 22
April 2017: Tue. 4, Wed. 5, Tue. 11, Wed. 12, Wed. 26

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UNIVERSITÀ CATTOLICA del Sacro Cuore

TOPICS IN SYMPLECTIC AND MULTISYMPLECTIC GEOMETRY

Ph.D. Course

Prof. M. Spera (UCSC - Brescia)

Lecture I

Prologue: Lagrange & Hamilton equations and reinterpretation of the formalism

Newton's equation (★) $\boxed{m\ddot{q} = -\frac{\partial V}{\partial q} = f}$ (conservative force)

$V = V(q)$ potential energy

$(q \equiv (q^i)_{i=1..n}$

$n = \#$ degrees of freedom

can be recovered via the Lagrange equations

(◆) $$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0}$$

$$\mathcal{L} = \mathcal{L}(q, \dot{q}, \cancel{t})$$

configuration space, velocity

$\mathcal{L} : TQ \to \mathbb{R}$

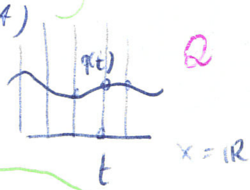
tangent bundle

for simplicity

Lagrangian

$$\boxed{\mathcal{L} = \underbrace{\frac{1}{2} m \dot{q}^2}_{\text{kinetic energy}} - \underbrace{V(q)}_{\text{potential energy}}}$$

[$n = 1$ m = mass of a pt particle (constant)]



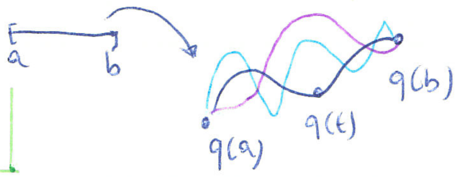
$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \quad \text{(linear momentum)}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = m\ddot{q} \qquad \frac{\partial \mathcal{L}}{\partial q} = -\frac{\partial V}{\partial q} = f$$

, so from (◆) we get (★)

★★★ Hamilton principle

(◆) come from looking at the "critical points" (curves) of the action functional



action → $$\boxed{S(q(t)) = \int_a^b \mathcal{L}(q(t), \dot{q}(t))\,dt}$$

we mean the curve $q = q(t)$

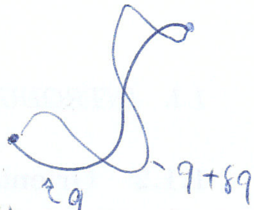
$(q = q(t),\ t \in [a,b]$, at least of class $C^2)$

with $q(a) = q(b)$ (fixed endpoints)

Variation:

$$\underbrace{(q+\varepsilon\delta q)}_{q_\varepsilon(t)}(t) = q(t)+\varepsilon\delta q(t)$$

$$\delta q(a) = \delta q(b) = 0$$



more generally take any $q_\varepsilon(t)$ with $q_0(t)=q(t)$
$\delta q(t) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} q_\varepsilon(t)$

Look at extremals

$$\boxed{dS(t)\cdot\delta q(t) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\big(S(\underbrace{q(t)+\varepsilon\delta q(t)}_{q_\varepsilon(t)})\big) = 0}$$

$(dS, \delta q)$

or, better δS

standard calculation "compute to 1st order in ε"

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\big(S(q(t)+\varepsilon\delta q(t))\big) = \int_a^b \Big(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\underbrace{\delta\dot q}_{=\dot{\delta q}}\Big)dt$$

notice this

integration by parts

$$= \int_a^b \Big(\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q}\Big)\Big)\delta q + \underbrace{\frac{\partial L}{\partial \dot q}\delta q\Big|_a^b}_{=0} = 0$$

$\forall\, \delta q$

since $\delta q(a) = \delta q(b) = 0$

$\Rightarrow$ (lemma of du Bois-Reymond) "fundamental lemma of calculus of variations" get Lagrange

$$\boxed{\frac{\partial L}{\partial q^i} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q^i}\Big) = 0}$$

2nd order $\boxed{\text{ODE}}$

$$\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^j\partial \dot q^i}\dot q^j - \frac{\partial^2 L}{\partial \dot q^j\partial \dot q^i}\ddot q^j = 0$$

non singular

$\leftrightarrow \ddot q = \dots$ ("Newton")

★ Field theoretic analogue

Einstein convention $\to$

$$\boxed{\partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu\varphi^i)}\Big) - \frac{\partial L}{\partial \varphi^i} = 0}$$

$\varphi = (\varphi^i(x))$ $x\in X$ spacetime

field (section of a fibre bundle $E\to X$ e.g. $E = X\times Q$

Now remove condition $\delta q(a) = \delta q(b) = 0$

we get a more general expression

$$d\mathcal{S}(q(t)) \cdot \delta q(t) = \int_a^b \delta q^i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^i} \right) dt + \frac{\partial L}{\partial \dot q^i}\delta q^i \Big|_a^b$$

"Euler-Lagrange operator"

get a 1-form

$$\equiv \int_a^b dt \cdot D_{EL} L(q, \dot q, \ddot q)\, \delta q + \Theta_L(q, \dot q) \cdot \hat{\delta q} \Big|_a^b$$

acting on "2nd order jets"

$$\Theta_L = \frac{\partial L}{\partial \dot q^i} dq^i$$

boundary part of $\delta \mathcal{S}$

Cartan (-Lagrange) form

$$\hat{\delta q} = \delta q \frac{\partial}{\partial q}$$

("jetification" of δq)

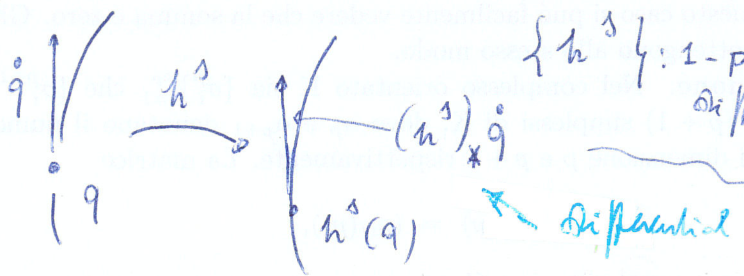
to be dealt with later on

$$\left(\frac{\partial L}{\partial \dot q} dq, \hat{\delta q} \right) = \frac{\partial L}{\partial \dot q}\delta q$$

*** Noether's theorem (standard approach)

" Any symmetry of L yields a conserved current "

$$(h^s q)(t) = q(s,t)$$



$\{h^s\}$: 1-parameter group of diffeomorphisms of Q

$\dot q(s,t)$

Assume $L(q(s,t), \dot q(s,t)) = L(q(t), \dot q(t))$ — invariance under h^s

(therefore $\frac{\partial L}{\partial s} = 0$) $\qquad q(t) \equiv q(0,t)$, $\dot q(t) \equiv \dot q(0,t)$

$L : TQ \to \mathbb{R}$

$L(h_* v) = L(v)$

Define the (Noether) current $\boxed{J = \frac{\partial L}{\partial \dot q} q'}$ $\qquad q' \equiv \left.\frac{\partial q(s,t)}{\partial s}\right|_{s=0}$

Then, along a solution of the L. equations, J is conserved, namely $\boxed{\frac{dJ}{dt} = 0}$

Indeed: derivatives commute

$$\frac{dJ}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) q' + \frac{\partial L}{\partial \dot q}\left(\frac{d}{dt} q'\right) = \frac{\partial L}{\partial q} q' + \frac{\partial L}{\partial \dot q}(\dot q)'$$

Lagrange

$$= \frac{\partial L}{\partial s}(q, \dot q) = 0$$

translation invariance $\to$ linear momentum

rotation invariance $\to$ angular momentum

Recall that if $L = L(q(t), \dot q(t), \cancel{t})$

(L does not explicitly depend on t)

one gets energy conservation:

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial \dot q}\ddot q = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\dot q + \frac{\partial L}{\partial \dot q}\ddot q = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}\dot q\right] \text{ ; therefore}$$

$$\frac{d}{dt}\Big[\underbrace{\frac{\partial L}{\partial \dot q}\dot q - L}_{E \text{ energy}}\Big] = 0 \Rightarrow E \text{ constant along trajectories}$$

{ check }

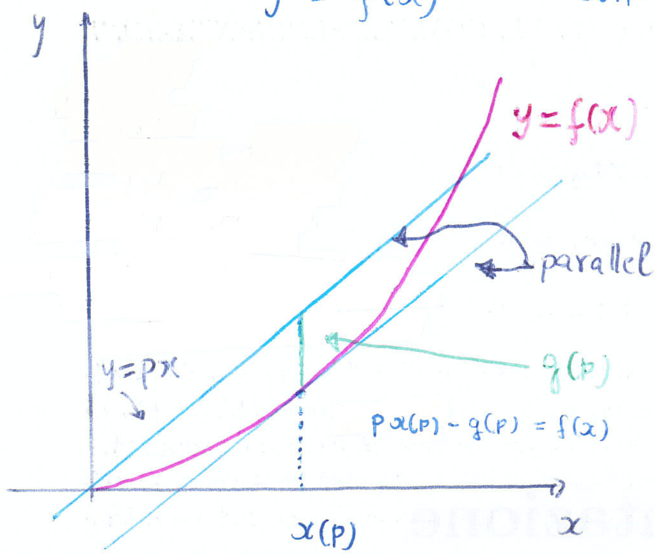
$L = \frac{1}{2}m\dot q^2 - V(q) \qquad \frac{\partial L}{\partial \dot q} = m\dot q$

$\frac{\partial L}{\partial \dot q}\dot q - L = m\dot q^2 - \frac{1}{2}m\dot q^2 + V(q)$

$= \frac{1}{2}m\dot q^2 + V(q) \equiv E$

$\boxed{\text{Aside}}$ **Legendre transform** (simplest case)

$y = f(x)$ convex $f'' > 0$



conjugate momentum (cotangent v.) velocity (tangent v.)

$$H(q,p,t) = p\dot q - L(q,\dot q,t)$$

Hamiltonian Lagrangian

$$p = \frac{\partial L}{\partial \dot q}$$ conjugate momentum to q

Legendre transform need

$$\frac{\partial^2 L}{\partial \dot q\,\partial \dot q} = m > 0 \quad \text{ok}$$

$$\left(\frac{\partial^2 L}{\partial \dot q_i\,\partial \dot q_j}\right) \text{ positive definite}$$

$$F(p,x) = px - f(x) \quad \leftarrow \text{maximise}$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow p - f'(x) = 0 \Rightarrow \text{get a unique } x = x(p)$$

The function $g(p) := F(p, x(p))$ is the Legendre transform of f

(with crystal clear geometrical meaning)

The Legendre transform is involutive : $f \mapsto g \longmapsto f$

$$G(x,p) = xp - g(p) \qquad g'(p) = x \Rightarrow p = p(x)$$

Obviously (see picture) $G(x, p(x)) = f(x)$

$$y = G(x,p) = xp - g(p) \quad \text{is a family of straight lines}$$

Let us determine its envelope:

$$\begin{cases} y - xp + g(p) = 0 \\ \frac{\partial}{\partial p}(y - xp + g(p)) = 0 \end{cases}$$

$$\begin{cases} y - xp + g(p) = 0 \\ x = g'(p) \end{cases} \Rightarrow y = x\cdot p(x) - g(p(x)) = f(x)$$

Thus, eventually

Legendre = envelope of the lines $y = px - f(x)$
(in the plane (p,y))

Examples:

for kinetic energy $(x \equiv q)$

$\boxed{f(x) = \frac{mx^2}{2}}$ $\quad f'(x) = mx \quad p = mx, \quad x = \frac{p}{m}$

$$\boxed{g(p)} = px - m\frac{x^2}{2} = \frac{p^2}{m} - \frac{m}{2}\frac{p^2}{m^2} = \boxed{\frac{p^2}{2m}}$$

$$f(x) = \frac{x^\alpha}{\alpha} \qquad g(p) = \frac{p^\beta}{\beta} \qquad x > 0$$

$\alpha, \beta > 1$

$\frac{1}{\alpha} + \frac{1}{\beta} = 1$

Notice that $\quad px - f(x) \le g(p)$

$\Rightarrow \quad \boxed{f(x) + g(p) \ge px} \quad$ (Young inequality)

In particular $\quad \boxed{px \le \frac{x^\alpha}{\alpha} + \frac{y^\beta}{\beta}}$

Hamilton's principle

(Hamiltonian form)

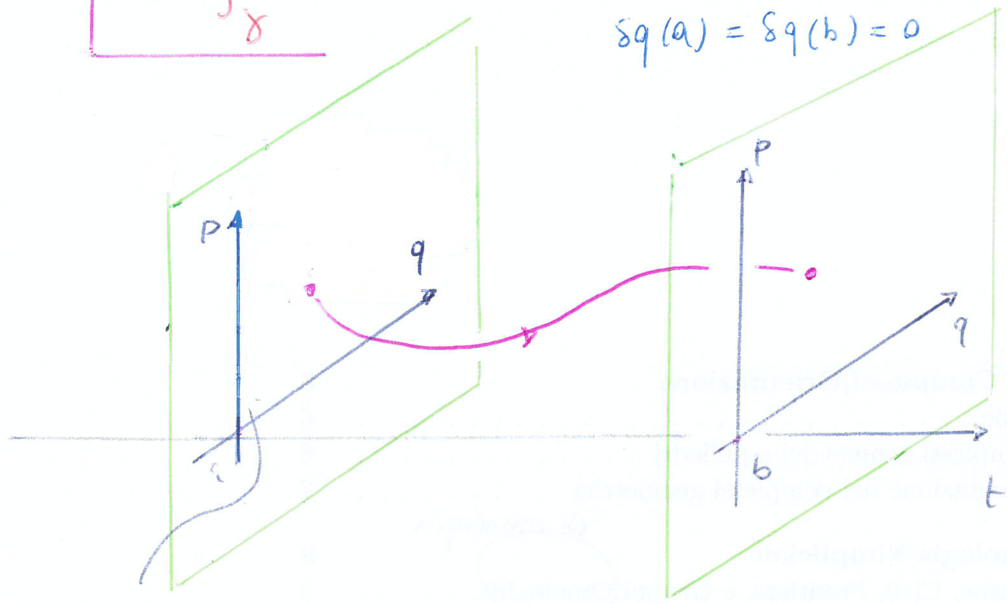
$$\boxed{\delta \int_\gamma p\,dq - H\,dt = 0}$$

$$L = p\dot{q} - H$$

$$p = \frac{\partial L}{\partial \dot{q}} \quad \text{(Legendre)}$$

conjugate momentum (to q)

$\delta q(a) = \delta q(b) = 0$



phase space

$\tilde{q} = q + \delta q$
$\tilde{p} = p + \delta p$

$\begin{cases} q = q(t) \\ p = p(t) \end{cases}$

Let us perform the calculation exploiting the variational principle again

varied curves: $\begin{cases} \tilde{q}(t) = q(t) + \varepsilon\,\delta q(t) \\ \tilde{p}(t) = p(t) + \varepsilon\,\delta p(t) \end{cases}$ $\qquad \delta q(a) = \delta q(b) = 0$

Then, expanding up to the first order

$$\int_\gamma \tilde{p}\,d\tilde{q} - \tilde{H}\,dt = \int_\gamma p\,dq - H\,dt + \varepsilon A$$

to be equated to 0

Let H do not depend explicitly on t

$$0 = A = \int_\gamma [\delta p\,dq + p\,d(\delta q) - \delta H\,dt] =$$

$$\int_a^b [\delta p\,\dot{q} + p\frac{d}{dt}\delta q - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p]\,dt$$

$$= \int_a^b \{(\dot{q} - \frac{\partial H}{\partial p})\delta p - (\dot{p} + \frac{\partial H}{\partial q})\delta q\}\,dt + \underbrace{p\,\delta q\Big|_a^b}_{=0}$$

arbitrary $\qquad$ $\delta q(a) = \delta q(b) = 0$

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\[2ex] \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

Hamilton equations

Again notice

$$\langle \delta S, v \rangle = \int_a^b D_H(v)\,dt + \Theta(\hat{v})\Big|_a^b$$

$\| \quad \begin{pmatrix}\delta q\\ \delta p\end{pmatrix}$

Hamilton operator

$\Theta = p\,dq$ <u>Cartan form</u>

$\hat{v} = \delta q \frac{\partial}{\partial q} + \delta p \frac{\partial}{\partial p}$

("jetification")

Let us reformulate Hamilton's equations in another guise

In the augmented phase space (q, p, t), set

$$X_H = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

and $\boxed{\Omega = dp \wedge dq - dH \wedge dt}$ $(= d(p\,dq - H\,dt))$ [presymplectic form in (q,p,t)]

Then $\boxed{\Omega(X, V) = 0 \quad \forall\, V \iff X = X_H}$ (★)

namely: $\underbrace{i_X \Omega}_{\text{contraction of } \Omega \text{ by } X} \equiv 0 \iff X = X_H$

$X = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial q}$

tangent to the curve $t \mapsto (t, q(t), p(t))$

Let us prove (★). So let $V = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + C\frac{\partial}{\partial p}$

compute $\Omega(X, V) =$

$$(dp \wedge dq)\left(\dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}, B\frac{\partial}{\partial q} + C\frac{\partial}{\partial p}\right) \quad ①$$

"area form"

$(dp \wedge dq)(z, w) = dp(z)\,dq(w) - dp(w)\,dq(z)$

← no terms in $\frac{\partial}{\partial t}$

$$- \Big[\underbrace{\Big(\frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp\Big)\wedge dt}_{\left(\frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp\right)\wedge dt}\Big]\Big(\underbrace{\frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}}_{X}, \underbrace{A\frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + C\frac{\partial}{\partial p}}_{V}\Big) \quad ②$$

$$① = \dot{p}B - \dot{q}C$$

$$\frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} \qquad A\frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + C\frac{\partial}{\partial p}$$

$$② = \left(-\frac{\partial H}{\partial q}\, dq \wedge dt\right)(X, V) - \frac{\partial H}{\partial p}\,(dp \wedge dt)(X, V)$$

$$= -\frac{\partial H}{\partial q}\left\{dq(X)\,dt(V) - dq(V)\,dt(X)\right\} - \frac{\partial H}{\partial p}\left\{dp(X)\,dt(V) - dp(V)\,dt(X)\right\}$$

$$= -\frac{\partial H}{\partial q}\left[\dot{q}A - B\right] - \frac{\partial H}{\partial p}\left[\dot{p}A - C\right]$$

$$\underbrace{① + ②}_{\substack{\parallel \\ 0 \\ \forall A, B, C}} = A\left[-\frac{\partial H}{\partial q}\dot{q} - \frac{\partial H}{\partial p}\dot{p}\right] + B\left[\dot{p} + \frac{\partial H}{\partial q}\right] + C\left[-\dot{q} + \frac{\partial H}{\partial p}\right]$$

$$\Rightarrow \boxed{\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}}$$

☆ Hamilton

automatically $= 0$

Also

$$-\frac{\partial H}{\partial q}\dot{q} - \frac{\partial H}{\partial p}\dot{p}$$

$$= -\frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0$$

Generalization: Hamilton - Volterra - De Donder - Weyl ☆☆☆

Lagrange: $\boxed{\partial_\mu\left(\dfrac{\partial \mathcal{L}}{\partial \varphi^i_{,\mu}}\right) - \dfrac{\partial \mathcal{L}}{\partial \varphi^i} = 0}$

$$\pi_i^{\ \mu} = \frac{\partial \mathcal{L}}{\partial_\mu \varphi^i}$$

conjugate momenta ☆☆

$$\varphi^i \leadsto \pi_i^{\ \mu}$$

$$\Rightarrow \partial_\mu \pi_i^{\ \mu} - \frac{\partial \mathcal{L}}{\partial \varphi_i} = 0$$

$$\mathcal{H} = \pi_i^{\ \mu}\varphi^i_{,\mu} - \mathcal{L}(\varphi^i, \partial_\mu\varphi^i)$$

$$\frac{\partial \mathcal{H}}{\partial \pi_i^{\ \mu}} = \varphi^i_{,\mu}$$

$$\frac{\partial \mathcal{H}}{\partial \varphi^i} = -\frac{\partial \mathcal{L}}{\partial \varphi_i}$$

$$\boxed{\begin{cases} \partial_\mu \pi_i^{\ \mu} = -\dfrac{\partial \mathcal{H}}{\partial \varphi^i} \\ \partial_\mu \varphi^i = \dfrac{\partial \mathcal{H}}{\partial \pi^\mu_{\ i}} \end{cases}}$$

Also notice ("Lie derivative")

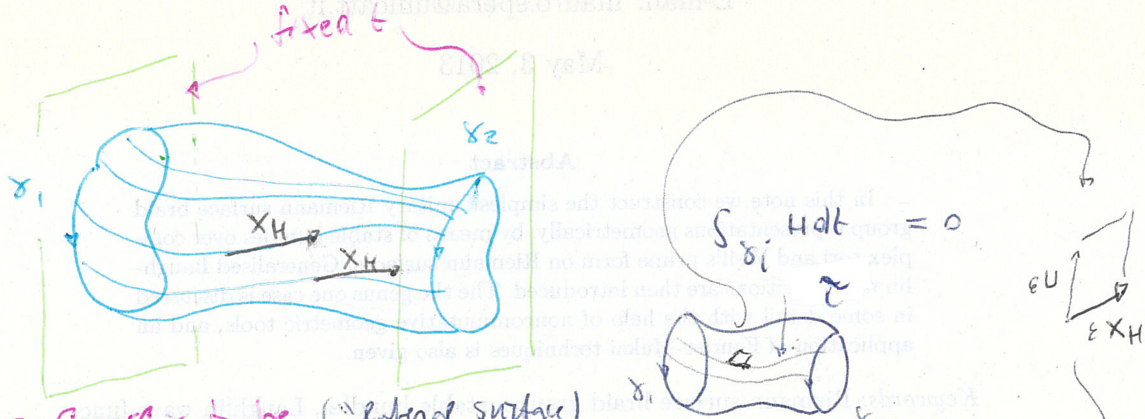
$$\boxed{\begin{aligned} \omega &= f\, dx_I \\ d\omega &= df \wedge dx_I \end{aligned}}$$

$$\frac{d}{dt}(dp\wedge dq) = d\dot p \wedge dq + dp \wedge d\dot q$$

$$= d\left(-\frac{\partial H}{\partial q}\right)\wedge dq + dp\wedge d\left(\frac{\partial H}{\partial p}\right)$$

$$= -\frac{\partial^2 H}{\partial p\,\partial q}\, dp\wedge dq + \frac{\partial^2 H}{\partial q\,\partial p}\, dp\wedge dq = 0 \quad \text{(Schwarz)}$$

Symplecticity



characteristic tube ("lateral surface")

$$\oint_{\gamma_2} p\,dq - \oint_{\gamma_1} p\,dq = \oint_{\gamma_2}(p\,dq - H\,dt) - \oint_{\gamma_1}(p\,dq - H\,dt)$$

"area element" of τ

$$= \oint_{\gamma_2-\gamma_1}(p\,dq - H\,dt) \overset{\text{Stokes}}{=} \int_\tau \overbrace{dp\wedge dq - H\,dt}^{\Omega} = 0$$

(since $\Omega(X_H, \cdot) = 0$)

$$\Rightarrow \oint_{\gamma_2} p\,dq = \oint_{\gamma_1} p\,dq$$ "invariance of circulation"

Furthermore $$\int_{D_1} dp\wedge dq = \int_{D_2} dp\wedge dq$$ "conservation of vorticity"

(Symplecticity again)

[Poincaré – Cartan $\mathcal{L}_X\, \omega^k = 0$]

in particular: Liouville

TOPICS IN SYMPLECTIC AND MULTISYMPLECTIC GEOMETRY

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Lecture II

Symplectic manifolds

- symplectic vector spaces
- symplectic manifolds
- Darboux - Weinstein theorem (statement, consequences, preparations for the proof - à la Moser)

* Symplectic vector spaces

V : m-dimensional vector space (over $\mathbb{R}$)

$\Omega : V \times V \to \mathbb{R}$ bilinear map

Ω skew-symmetric : $\Omega(u,v) = -\Omega(v,u) \quad \forall\, u,v \in \mathbb{R}$

Then Ω can be put in standard form upon finding a basis $(u_1 \ldots u_k, e_1 \ldots e_n, f_1 \ldots f_n)$ such that (so $k + 2n = m$)

$$\begin{cases} \Omega(u_i, v) = 0 & i = 1, \ldots k, \quad \forall\, v \in V \\ \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 & i,j = 1 \ldots n \\ \Omega(e_i, f_j) = \delta_{ij} & i,j = 1 \ldots n \end{cases}$$

called a canonical basis

matrix form: $\Omega(u,v) = \boxed{ u^T } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{bmatrix} \boxed{v}$

Proof: adapt the Gram-Schmidt process

let $U = \{ u \in V \mid \Omega(u,v) = 0 \;\; \forall\, v \in V \}$.

let $(u_1 \ldots u_k)$ be a basis of U and let $V = U \oplus W$ (complement)

[if $k = m$ we are finished]

Let $e_1 \in \tilde{W}$ $e_1 \neq \underline{0}$. Then $\exists\ f_1 \in \tilde{W}$ with

$\Omega(e_1, f_1) \neq 0$ (assume w.l.o.g. $\Omega(e_1, f_1) = 1$)

Let $W_1 = \langle e_1, f_1 \rangle$

$$W_1^{\Omega} = \{ w \in \tilde{W} \,/\, \Omega(w, v) = 0\ \forall\ v \in W_1 \}$$

W_1 and W_1^{Ω} are in direct sum: $W_1 \cap W_1^{\Omega} = \{0\}$

Indeed let $v = a e_1 + b f_1 \in W_1 \cap W_1^{\Omega}$ $\quad \Omega(v, u) = 0\ \forall\ u \in W_1$

then $\underset{0}{\underset{\shortparallel}{\Omega(v, e_1) = -b}}$, $\underset{0}{\underset{\shortparallel}{\Omega(v, f_1) = a}}$

$\Rightarrow a = b = 0$ (i.e. $v = \underline{0}$)

Also, $\boxed{W = W_1 \oplus W_1^{\Omega}}$ Take $v \in W$; then

let $\Omega(v, e_1) = c$, $\Omega(v, f_1) = d$

Then

$$v = \underbrace{-cf_1 + de_1}_{\in\, W_1} + \underbrace{v + cf_1 - de_1}_{\in\, W_1^{\Omega}}$$

quick check
$\Omega(v + cf_1 - de_1, e_1)$
$= c + c\,\underbrace{\Omega(f_1, e_1)}_{-1}$
$= c - c = 0$
similarly for f_1 ...

If $W_1^{\Omega} = \{0\}$ we are done; otherwise let $e_2 \in W_1^{\Omega}$,

$e_2 \neq \underline{0}$, take $f_2 \in W_1^{\Omega}$ such that $\Omega(e_2, f_2) = 1$

Let $W_2 = \langle e_2, f_2 \rangle$ and go on. Eventually we have

work in $\tilde{W}$

$$V = U \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

mutually Ω-orthogonal $\quad W_i = \langle e_i, f_i \rangle$

$\Omega(e_i, f_i) = 1$

$k = \dim U$ does not depend on the choice of a basis

$\Rightarrow$ $2n = m - k$ is an invariant of (V, Ω), the

rank of Ω.

Define $\tilde{\Omega} : V \longrightarrow V^*$ ← dual of V

$$\underbrace{\tilde{\Omega}(v)}_{V^*}(\underset{V}{u}) := \Omega(v, u)$$

then $\ker \tilde{\Omega} = U$

Ω is called symplectic (or non degenerate) if $\tilde{\Omega}$ is bijective (i.e. $\ker \tilde{\Omega} = U = \{0\}$) by N+R theorem

then

Ω: linear symplectic structure
(V, Ω): symplectic vector space

Clearly, in this case $\tilde{\Omega} : V \xrightarrow{\simeq} V^*$ bijective

$k = 0 \Rightarrow m = \dim V = 2n$ (even)

$\exists$ basis $(e_1 - e_n, f_1 - f_n)$ with $\begin{cases} \Omega(e_i, f_j) = \delta_{ij} \\ \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 \end{cases}$

symplectic basis

$$\Omega(u, v) = u^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} v$$

matrix form

Let $\Omega = -\Omega^T$ a skew-symmetric $n \times n$ real matrix. Then
$\det(\Omega) = \det(-\Omega^T)$
$= (-1)^n \det(\Omega^T)$
$= (-1)^n \det \Omega$
$\Rightarrow \Omega$ is non degenerate
$\Leftrightarrow n$ even

prototype $(\mathbb{R}^{2n}, \Omega_0)$

basis vectors: $e_i = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leftarrow i \qquad f_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow n+i$

★ symplectomorphism: $\varphi : (V, \Omega) \to (V', \Omega')$

bijective linear map $\varphi : V \to V'$ such that $\varphi^* \Omega' = \Omega$ pull-back

$$(\varphi^* \Omega')(u, v) := \Omega'(\underbrace{\varphi(u), \varphi(v)}_{V'})$$

$((V, \Omega), (V', \Omega'))$ symplectomorphic.

$\star\star$ Therefore, any two symplectic vector spaces having the same dimension are symplectomorphic

Let $W \leq V$ (symplectic vector space)

- W isotropic : $W \leq W^{\Omega}$

  i.e $\Omega(w_1, w_2) = 0 \quad \forall\, w_1, w_2 \in W$

- W coisotropic $W^{\Omega} \leq W$

- W Lagrangian isotropic & coisotropic

  i.e $W^{\Omega} = W$

$\star$ W is Lagrangian $\iff$ W is maximally isotropic

i.e $\dim W = n = \frac{1}{2} \dim V$

Examples : $\langle e_1 \dots e_n \rangle$, $\langle f_1 \dots f_n \rangle$ are Lagrangian sub spaces

For example, any codimension 1 subspace of V is coisotropic

ex: $W = \langle e_2\, e_3 \dots e_n, f_1 \dots f_n \rangle$ $\quad W^{\Omega} = \{ x \mid \Omega(x, e_i) = \Omega(x, f_j) = 0$

$i = 2 \dots n,\ j = 1 \dots n \} = \langle f_1 \rangle$

$\star$ symplectic group $\equiv O(V, \Omega)$ orthogonality with respect to Ω

$$:= \{ A \in GL(V) \mid \Omega(Ax, Av) = \Omega(x, v) \ \forall\, x, v \in V \}$$

In numerical terms (with respect to a generic basis)

$$(Ax)^T \Omega A v = x^T \cdot A^T \Omega A \cdot v = x^T \cdot \Omega \cdot v \qquad \forall\, x, v$$

namely $\boxed{A^T \Omega A = \Omega}$ or, equivalently $A^T \Omega = \Omega A^{-1}$

$A^{-1} = \Omega^{-1} A^T \Omega$

If $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ $\quad \Omega^{-1} = -\Omega$ $\qquad$ Notation: $Sp(V, \mathbb{R})$

★ <u>symplectic manifold</u> M manifold equipped with a closed, non degenerate 2-form ω

$d\omega = 0$

exterior differential

$\omega_p : T_pM \times T_pM \to \mathbb{R}$

M

skew-symmetric, non degenerate $\forall p \in M$ (ω_p symplectic) (and smoothly varying). Obviously $\dim M = 2n$ (even)

$\boxed{\omega : \text{symplectic form}}$

<u>Basic example</u> (and local model for all f.d. symplectic manifolds, by the Darboux (-Weinstein) theorem

i.e. a symplectic vector space viewed as a manifold

$M = \mathbb{R}^{2n}$, coordinates $(q^1 - q^n, p_1 - p_n)$

$$\boxed{\omega_0 = \sum_{i=1}^{n} dq^i \wedge dp_i \equiv dq^i \wedge dp_i}$$

<u>Symplectic basis</u>: $\left(\left.\frac{\partial}{\partial q^i}\right|_m , \left.\frac{\partial}{\partial p_i}\right|_m \right)$

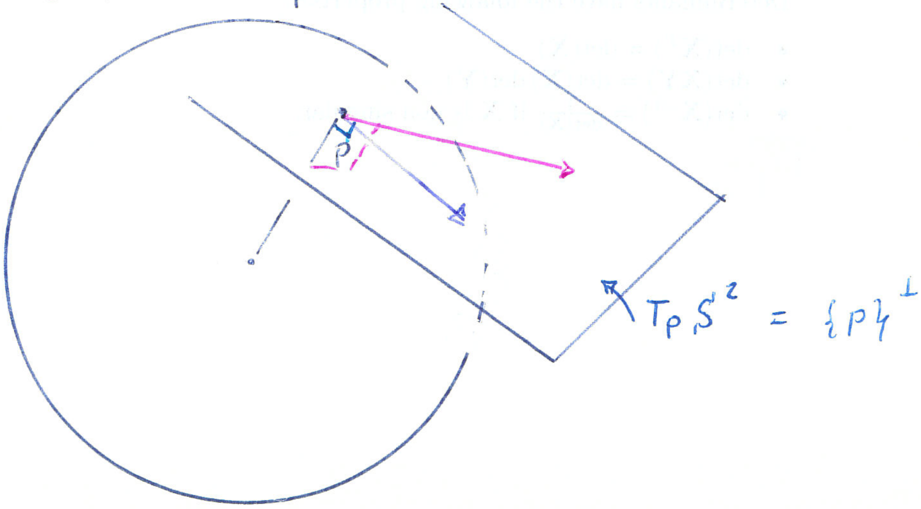
$m \in M$

in <u>complex guise</u>: $M = \mathbb{C}^n \cong \mathbb{R}^n$, coordinates $z_k = x_k + i\, y_k$ ($q^k + i\, p_k$) $k = 1..n$

$$\boxed{\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k}$$

<u>Important example</u>

$S^2 \equiv$ unit vectors in $\mathbb{R}^3$ $\quad T_pS^2 \equiv$ vectors $\perp p$



"mixed" product

$$\omega_p(u, v) := \langle p, u \times v \rangle \qquad T_pS^2 = \{p\}^\perp$$

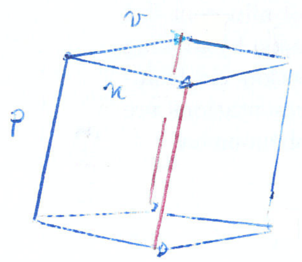
$u, v \in T_pS^2$, $\langle p, u \times v \rangle = \det\begin{pmatrix} | & | & | \\ p & u & v \\ | & | & | \end{pmatrix}$

ω is closed (top degree)

ω_p is non degenerate : $u \times v \,/\!/\, p$ and

$\omega_p(u, v) =$ oriented area of the parallelogram spanned by u and v = oriented volume of the parallelepiped spanned be p, u, v

$= 0 \quad \forall v \iff u = 0$



Remark

Einstein convention

Given $\omega \equiv \frac{1}{2} \omega_{ij}\, dx^i \wedge dx^j$

notice this

The associated matrix is

$$\Omega = (\omega_{ij}) \quad (\Omega^T = -\Omega, \ \omega_{ij} = -\omega_{ji})$$

example $\quad dq \wedge dp = \frac{1}{2} dq \wedge dp - \frac{1}{2} dp \wedge dq$

$$\Rightarrow \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Moreover, if $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (canonical form), the equation

$$\Omega x = y$$

(which can be solved for any y), yields

$$\boxed{x = \Omega^{-1} y = (-\Omega)\, y = \Omega^T y}$$

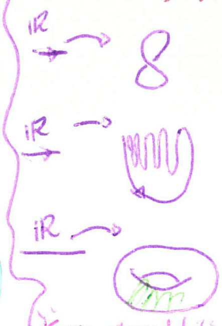
dare of check

The Darboux – Weinstein theorem (à la Moser)

(See Guillemin-Sternberg STP 1984)

X manifold $\quad Y \hookrightarrow X$ embedded submanifold

$Y \xrightarrow{i} X$ embedding: i injective immersion (i injective, i_* injective) + $i: Y \to i(Y) \subset X$ homeomorphism, with $i(Y)$ equipped with the relative topology induced by X



Kronecker's foliations are not embeddings

σ form on X $\quad \sigma|_Y$ restriction to Y (it can be evaluated on non tangent vectors to Y...)

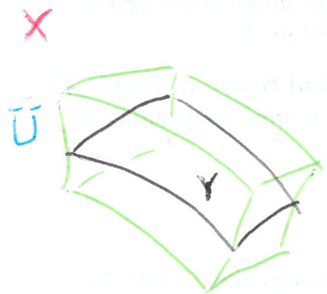
ω_0, ω_1 non singular closed 2-forms on X such that

$$\boxed{\omega_0|_Y = \omega_1|_Y}$$

Then, there exists a neighbourhood $U \supset Y$ such that and a diffeomorphism $f: U \to X$

(i) $\quad f(y) = y \quad \forall y \in Y$

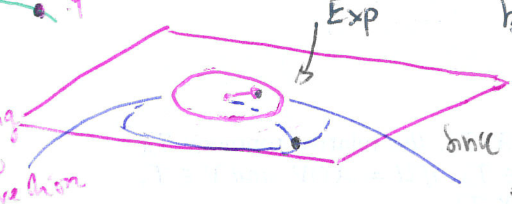
(ii) $\quad f^* \omega_1 = \omega_0$



Corollary: if $Y = \{pt\}$, then two forms agreeing on the tangent space to the point coincide in a neighbourhood of it up to a diffeomorphism

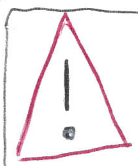
$\|v\| = 1$

geodesic emanating from p, with direction $v = \frac{\sigma'}{\|\sigma'\|}$



Using the exponential map with respect to some Riemannian metric, one can find a diffeomorphism between a neighbourhood of $0 \in T_p X$ and a neighbourhood $U \ni p$ in X. Since any two symplectic linear forms agree up to a linear map, the theorem yields the sought for local coordinate system

[far reaching generalizations are possible]



This result is not valid for Riemannian metrics: a metric is locally euclidean $\Leftrightarrow$ the Riemann curvature tensor vanishes

More clearly, given ω and a coord. system $\{x^i, s_i\}$ for which $\omega = dx^i \wedge ds_i$ at p, set $\omega' := dx^i \wedge ds_i$ and, given the above f, define $q^i = x^i \circ f$, $p_i = s_i \circ f$: then $\omega = dq^i \wedge dp_i = f^* \omega'$ $\quad \omega = \omega'$ at p

◇ Aside

Before delving into the proof, let us make a couple of technical digression

On the Lie derivative of differential forms

$$\mathcal{L}_X\omega = \frac{d}{dt}(F_t^X)^*\omega$$

F_t^X: flow of X

Recall that, for 1-forms ω $(X, Y \in \mathfrak{X}(M))$

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

w.l.o.g. this can be easily checked for $\omega = u\,dv$ "semilocal formalism"

Now compute

"Leibniz"

$$\underset{\Lambda^0}{\underset{\cap}{X[\omega(Y)]}} = \mathcal{L}_X(\omega(Y)) = (\mathcal{L}_X\omega)(Y) + \omega(\mathcal{L}_X Y)$$

arguments

$$= (\mathcal{L}_X\omega)(Y) + \omega([X,Y])$$

$$\mathcal{L}_X\omega(Y) = X\,\omega(Y) - \omega([X,Y]) = d\omega(X,Y) + Y\omega(X)$$

but $\quad Y\cdot\omega(X) = (d\omega(X), Y) \equiv d\omega(X)(Y)$,

thus

contraction

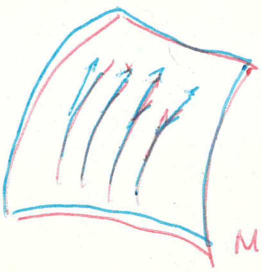
$$\mathcal{L}_X\omega(Y) = i_X\, d\omega(Y) + d(i_X\omega)(Y)$$

i.e.

$$(\diamond) \qquad \boxed{\mathcal{L}_X = i_X d + d\, i_X}$$

*** Cartan magic formula

The whole argument can be extended to k-forms



If $\{F_t^X\}$ is the flow of X on M, then the very definition of $\mathcal{L}_X$ and $(\diamond)$ yield (notational abuse)

$$F_1^*\omega - \underset{\omega}{\underset{\parallel}{F_0^*\omega}} = \int_0^1 i_X d\omega + d\, i_X\omega$$

"fibre integration"

(generalizing $F(b)-F(a) = \int_a^b F'(t)\,dt$)

Also, if ω depends on t as well, we have

$$\frac{d\omega_t}{dt} = \frac{\partial\omega_t}{\partial t} + \mathcal{L}_X\omega_t = \frac{\partial\omega_t}{\partial t} + d\, i_X\omega_t + i_X d\omega_t$$

"total, or material derivative"

A further

★ Technical digression

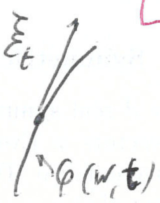
$$W \xrightarrow{\varphi_t} Z \qquad \{\varphi_t\} \quad \text{smooth 1-parameter \underline{family} of maps}$$

$$\varphi : W \times I \longrightarrow Z \quad \text{smooth}$$
$$(w, t) \longmapsto \varphi_t(w)$$

in general, if $\xi \in \mathcal{X}(W)$, $\varphi_* \xi$ (push-forward at any pt) is not a v. field on Z.
No problem for φ^* of forms

ξ_t = tangent field along φ_t

$$\xi_t : W \longrightarrow TZ$$
$$w \longmapsto \xi_t(w), \text{ tangent to } t \mapsto \varphi_t(w)$$



⚠ ξ_t is not a v. f. on Z

If $\sigma \in \Lambda^{k+1}(Z)$, the form

$$\varphi_t^*(i(\xi)\sigma) \in \Lambda^k(W)$$

contraction

is well-defined: ⚠ this is not a form

$$[\varphi_t^* \, i(\xi)\sigma](\eta_1 - \eta_k) = [i(\xi)\sigma]((\varphi_t)_*(\eta_1) - (\varphi_t)_*(\eta_k))$$

If σ_t is a smooth 1-par. family of forms,

$\varphi_t^* \sigma_t$ is again a smooth 1-par. family of forms

and explicit dependence on t — notice in r-s $\frac{d\sigma_t}{dt}$ — well-defined — well-defined

$$(★) \qquad \boxed{\frac{d}{dt}\varphi_t^*\sigma_t = \varphi_t^* \frac{\partial \sigma_t}{\partial t} + \varphi_t^*\left(i(\xi_t)\, d\sigma_t\right) + d\left[\varphi_t^* \, i(\xi_t)\sigma_t\right]}$$

can be checked by a local computation

Notice that if $W = Z$, σ is constant; ξ is a v. field and φ_t is its flow, then

$$\frac{d}{dt}\varphi_t^*\sigma \Big|_{t=0} = \mathcal{L}_\xi \sigma = i_\xi d\sigma + d\, i_\xi \sigma$$

Lie derivative $\qquad$ $\boxed{\text{CARTAN!}}$

See Guillemin - Sternberg STP 1984

Lecture III

Back to proof of Darboux

Let $Y \subset X$ embedded

- Proof of Darboux-Weinstein
- Symplectic structure of T^*Q
- Hamiltonian mechanics
- Homogeneous spaces

Assume existence of a smooth retraction

$$\varphi : X \xrightarrow{\text{onto}} Y$$

i.e.

φ_t smooth family of maps $X \to X$

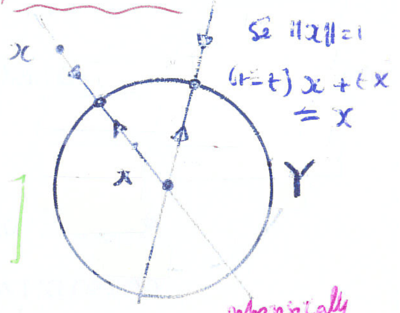
$\varphi_0 = \varphi : X \to Y \qquad \varphi_1 = id$

$\varphi_t y = y \quad \forall y \in Y, \forall t$

Example $n \geq 2$

$\mathbb{R}^n \setminus \{0\} \to S^n$

$\varphi_t : x \longmapsto (1-t)\dfrac{x}{\|x\|} + t x$



[If X is a vector bundle (or convex neigh. of the zero section), Y the zero section, then multiplication by t does the job]

Using Exp on the normal bundle to Y $\left(NY = TX/TY \right)$ intrinsically

with respect to a Riemannian metric

We can arrange that we have a retraction $U \to Y$
"tubular neighbourhood theorem"

Then, $\forall\, \sigma$ on X and on some U

$$\boxed{\sigma - \varphi_0^* \sigma} = \int_0^1 \frac{d}{dt}(\varphi_t^* \sigma)\, dt$$

$(\mathcal{S} d = d \mathcal{S})$

$$= \int_0^1 \varphi_t^*(i(\xi_t)\, d\sigma)\, dt + d \int_0^1 \varphi_t^*(i(\xi_t)\sigma)\, dt$$

get an obvious operator I:

$$\equiv I\, d\sigma + \boxed{d\, I\sigma}$$

$$I : \Lambda^k(X) \to \Lambda^{k-1}(X)$$

"homotopy operator"

$$\boxed{\sigma - \varphi_0^* \sigma = d I \sigma + I\, d\sigma}$$

Now set $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t\underset{\omega_1 - \omega_0}{\overset{\shortparallel}{\sigma}}$

$\sigma|_Y = 0 \implies \varphi_0^* \sigma = 0, \quad d\sigma = 0$

$\implies \sigma = d(\underset{\beta}{I\sigma}) \qquad \beta \equiv I\sigma \qquad \beta|_Y = 0$

but $\omega_t|_Y = \omega_0|_Y = \omega_1|_Y \implies$ $\omega_t|_Y$ is non degenerate

$\implies$ locally (on some U) ω_t stays non degenerate $\forall\ 0 \leq t \leq 1$

Then find η_t (v. field)

such that $\boxed{i(\eta_t)\,\omega_t = -\beta}$ ← ★ This is a most used argument

η_t is <u>unique</u> (non degeneracy of ω_t)

$\Rightarrow$ integrate it to f_t $\quad (f_t|_Y = id)$

one can arrange so that f_t is dif. for $0 \le t \le 1$

$f_0 = id \qquad \frac{d}{dt}\omega_t = \sigma$

We have $\qquad (d\omega_t = 0)$

$$f_1^* \omega_1 - \omega_0 = \int_0^1 \frac{d}{dt}(f_t^* \omega_t)\, dt =$$

$$\int_0^1 f_t^* (\underbrace{\sigma}_{d\beta} + d(\underbrace{i(\eta_t)\omega_t}_{-\beta}))\, dt = 0$$

$$d\beta - d\beta$$

$\Rightarrow$ ★ f_1 is the <u>desired</u> <u>diffeomorphism</u>

□

★ <u>Bonus</u> : The same technique can be employed to prove the <u>Poincaré</u> <u>lemma</u>

"a closed k-form is locally exact, for $k \ge 1$"

Again $Y = \{pt\}$ $\quad \varphi_0 \sigma = 0$ for $k \ge 1$ ...

This technique is also called "Moser trick".

** T^*Q (cotangent bundle of Q) as a symplectic manifold

$\pi: T^*Q \to Q$ $\qquad \pi_*: T(T^*Q) \to TQ$

$(q^i, p_i) \mapsto (q^i)$

$(q, p) \mapsto q$

Tautological 1-form θ

$$\boxed{\theta_{q,p} := \pi^* p}$$

$\pi^*: T_q^*Q \to T^*_{(q,p)}(T^*Q)$

$\boxed{\pi_* \text{ surjective} \Rightarrow \pi^* \text{ injective}}$

$\pi^*(\omega_1 - \omega_2) = 0$ means

$\pi^*(\omega_1-\omega_2)(v) = 0 \quad \forall v \in TE$

$(\omega_1 - \omega_2)(\pi_* v) = 0 \quad \forall v \in TE$

$(\omega_1-\omega_2)(w) = 0 \quad \forall w \in TB$

$\omega_1 - \omega_2 = 0$, $\omega_1 = \omega_2$

explicitly:

$F^*p(X) = p(F_*(X))$

$$\boxed{\underset{\substack{\cap \\ T_{q,p}(T^*Q)}}{\theta_{q,p}(X)} = p(\underset{\substack{\cap \\ T_qQ}}{\pi_*(X)})}$$

θ is smooth, and $\boxed{\omega = -d\theta}$ is a symplectic form

In coordinates: $\boxed{p = p_i\, dq^i}$ $\qquad \pi: (q,p) \mapsto q$

"symplectic potential" for ω

$$\boxed{\theta_{(q,p)} = \pi^* p = p_i\, dq^i}$$

summation understood

(i.e. p itself) whence the name "tautological"

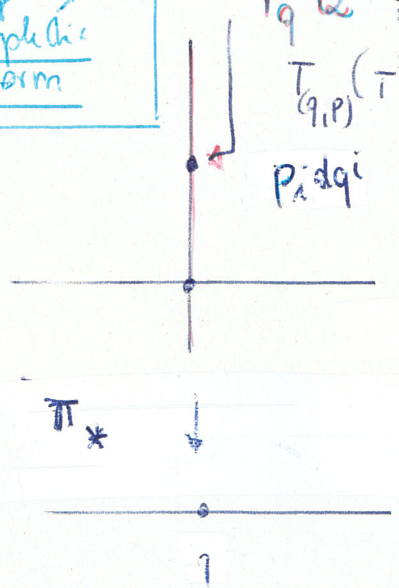
(Smoothness is clear) $\qquad \omega$ is closed (in fact it is exact)

$$\boxed{\omega = -d\theta = dq^i \wedge dp_i}$$

** Symplectic form

$T_q^*Q \equiv VT^*Q$ (vertical part of $T(T^*Q)$)

$T_{(q,p)}(T^*Q)$



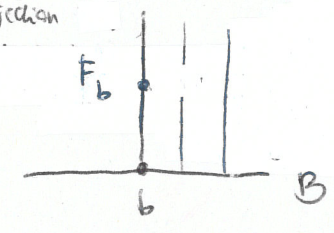
$T_qQ \dashrightarrow \mathbb{R}$

$\theta_{q,p}$

(locally trivial) fibre bundle

$\pi: E \xrightarrow{F} B$

projection

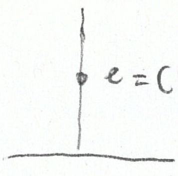


E: total space

B: base

π: projection

$F_b = \pi^{-1}(b) \simeq F$: typical fibre



$(VE)_e := T_e F_b \leq T_e E$

$VE \to B$

vertical bundle

Aside

★ geometric interpretation of closure for 1-forms

Notice that if $\sigma : M \to T^*M$ is a 1-form

then

$$\boxed{d\sigma = 0 \iff \sigma(M) \subset T^*M \text{ is a Lagrangian embedding}^{(+)}}$$

in fact $\sigma(x^i) = (x^i, \sigma_i(x))$ $\qquad \sigma = \sigma_i dx^i$

$\sigma(M)$ is Lagrangian (max. isotropic)

iff it is isotropic ($\sigma(M)$ has dim n)

i.e. $\sigma^* \omega = 0$

new notation ...

but $\boxed{\sigma^* \theta = \sigma^* (\xi_i dx^i) = \sigma_i dx^i = \sigma}$ (+)

(+) $F : M \to N$ injective immersion.
F is a smooth embedding if one of the following conditions hold
(a) M compact
(b) F proper $\quad F^{-1}(K)$ = K' (compact)

A map with a left inverse is proper

σ is a section $\quad \pi \cdot \sigma = i$

See eq. Lee: Smooth Manifolds

Therefore

$$\sigma^* \omega = -\sigma^* d\theta = -d(\sigma^* \theta)$$
$$= -d\sigma$$

$$\Rightarrow \boxed{\sigma(M) \text{ Lagrangian} \iff d\sigma = 0}$$

$\pi \circ \sigma = id$

(+) More invariantly: $(\sigma^*\theta)_p(X) = \theta_{\sigma(p)}(\sigma_*(X)) = \sigma_p(\pi_* \sigma_*(X))$

$\pi_* \sigma_* = (\pi \circ \sigma)_* = id$ (chain rule)

$$= \sigma_*(X) \quad \Rightarrow \boxed{\sigma^* \theta = \sigma}$$

★ Hamiltonian mechanics

(M, ω, H) Hamiltonian system

symplectic manifold — Hamiltonian: any "privileged" smooth function

Given $X \in \mathfrak{X}(M)$, consider its flow F_t^X

(if X is complete, the flow is defined $\forall\, t \in \mathbb{R}$; this happens, in particular, if M is compact)

★ X is termed **symplectic vector field** if its flow preserves the symplectic form ω:

$$(F_t^X)^* \omega = \omega$$

Lie derivative

infinitesimally: (★) $\boxed{\mathcal{L}_X \omega = 0}$

clearly, given a 1-parameter group $\{\varphi_t\}$ of symplectomorphisms, its generator is a symplectic vector field

(★) becomes (Cartan)

$$0 = \mathcal{L}_X \omega = d\, i_X \omega + i_X \underbrace{d\omega}_{=0} = d(i_X \omega)$$

i.e. $\boxed{d(i_X \omega) = 0}$

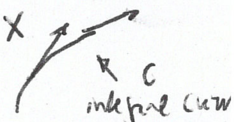
namely $i_X \omega$ closed

X is termed **hamiltonian** when $i_X \omega$ is exact:

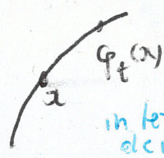
$\boxed{i_X \omega = d\lambda_X}$ ← Hamiltonian pertaining to X

[λ_X is determined up to a constant, if M is connected]

$\dot{c}(t) = X(c(t))$



$X(x)$ = velocity of the curve $\varphi_t(x)$ at x



in terms of derivations

$$X(x)(f) := \left.\frac{d}{dt}\right|_{t=0} f(\varphi_t(x))$$

Given $H \in \mathcal{C}^\infty(M)$, $\exists!\, X_H$ (hamiltonian v.f. pertaining to H)

Hamiltonian s.t.

$$\boxed{i_{X_H}\omega = dH}$$

(clear by non degeneracy of ω)

In matrix terms: $X_H^T \Omega\, v = dH \cdot v$

$X_H^T \Omega = dH$

$\Omega^T \cdot X_H = (dH)^T$

$X_H = \Omega^{-T} \cdot dH^T$

$X_H = (-\Omega)^{-1} \cdot dH^T$

$$\begin{pmatrix} | \\ | \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} | \\ \partial_1 H \\ | \end{pmatrix}$$

if $\Omega = \mathbb{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ $-\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ $(-\Omega)^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \mathbb{J}$

$\mathbb{J}^2 = -1$

* Poisson brackets

$$\boxed{\{f, g\} := \omega(X_f, X_g)}$$

f: hamiltonian (classical) observable

X_f, X_g: hamiltonian vector fields

$= (i_{X_f}\omega)(X_g)$

$= df(X_g) = X_g(f)$

$= -X_f(g)$

in particular, for (M, ω, H)

$\{f, H\} = -X_f(H) = X_H(f)$

and $\dot{f} = \dfrac{df}{dt} = \dfrac{\partial f}{\partial t} + \mathcal{L}_{X_H} f = \dfrac{\partial f}{\partial t} + X_H(f) = \dfrac{\partial f}{\partial t} - X_f(H) = \dfrac{\partial f}{\partial t} + \{f, H\}$

$$\boxed{\dot{f} \equiv \frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}}$$

* Hamilton equations

Check: $\quad \omega = dq \wedge dp$

X_q : $\quad i_{X_q}\omega = dq \qquad X_q = -\frac{\partial}{\partial p}$

$X_p \qquad i_{X_p}\omega = dp$

$$X_H = \xi \frac{\partial}{\partial q} + \eta \frac{\partial}{\partial p}$$

$$i_{X_H}\omega = dH$$

$$i_{X_H}(dq \wedge dp) = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp$$

$$\xi\, dp - \eta\, dq = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp$$

$$\xi = \frac{\partial H}{\partial p} \qquad \eta = -\frac{\partial H}{\partial q}$$

$$X_H = \frac{\partial H}{\partial p} \cdot \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p}$$

integral curves:

$$\boxed{\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}}$$ <u>Hamilton equations</u>

recovered

A symplectic, not hamiltonian vector field

Take $\left(M = \mathbb{R}^2 \setminus \{(0,0)\},\ \omega = dx \wedge dy\right)$

$$dx \wedge dy = r\,dr \wedge d\theta$$ ← polar coordinates

$$X := \frac{1}{r}\frac{\partial}{\partial r}$$

angle is not a function on $\mathbb{R}^2 \setminus \{(0,0)\}$!

Then $$i_X\,\omega = d\varphi$$ ← angular form

$$d\varphi = d \arctan \frac{y}{x} = \dots = \frac{x\,dy - y\,dx}{x^2+y^2}$$

a crucial object throughout mathematics

[one also checks directly that $\mathcal{L}_X \omega = 0$, via Leibniz rule]

Another example: $\left(M = \mathbb{T}^2 = S^1 \times S^1 \text{ torus},\ \omega = d\varphi \wedge d\psi\right)$ ← angular variables

$\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}$ are both symplectic but not hamiltonian

Clearly, if $H^1(M,\mathbb{R}) = 0$, then every symplectic vector field is hamiltonian. In the first example above $H^1(\mathbb{R} \setminus \{(0,0)\}) \cong \mathbb{R}$, whereas $H^1(\mathbb{T}^2, \mathbb{R}) \cong \mathbb{R}^2$

Let us check that

$$(\blacklozenge) \quad \boxed{X_{\{g,h\}} = -[X_g, X_h]}$$

in general, $X, Y \in \mathcal{X}^{symp} \Rightarrow [X,Y] \in \mathcal{X}^{ham}$

and $\boxed{\lambda_{[X,Y]} = \omega(Y,X)}$ (✱) (Sternberg)

indeed, from $\boxed{i_X \mathcal{L}_Y - \mathcal{L}_Y i_X = i_{[X,Y]}}$ ← check this

one has $$i_{[X,Y]}\omega = i_X \underbrace{\mathcal{L}_Y \omega}_{\substack{\shortparallel \\ 0 \\ Y \text{ symplectic}}} - \mathcal{L}_Y \underbrace{i_X \omega}_{\substack{\shortparallel \\ d\lambda_X}}$$

($\blacklozenge$) immediately follows from (✱)

$$= -d\, i_Y i_X \omega - i_Y \underbrace{d\, d\lambda_X}_{\substack{\shortparallel \\ 0}} = -d(\omega(X,Y))$$

$$= +d(\omega(Y,X))$$

$\Rightarrow$ (✱) follows

"Jacobiator"

$(\blacklozenge) \Rightarrow \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}$
$= c_{fgh} =$ constant

Let us now check that $\{\,,\,\}$ endows $\mathcal{C}^\infty(M)$ with a Lie algebra structure, whence the correspondence $f \mapsto X_f$ becomes an anti-homomorphism of Lie algebras

★ This is a consequence of the closure of ω

First recall that, for $\omega \in \Lambda^k(M)$

$$d\omega(X_1 \dots X_{k+1}) = \sum_{1\le i\le k+1} (-1)^{i-1} X_i[\omega(X_1 \dots \hat{X}_i \dots X_{k+1})]$$
$$+ \sum_{1\le i<j\le k+1} (-1)^{i+j}\, \omega([X_i, X_j], X_1, \dots \hat{X}_i \dots \hat{X}_j \dots X_{k+1})$$

omitted

✱ intrinsic formula for $d\omega$

Also, if $X, Y, Z \in \mathfrak{X}^{symp}(M)$ (ω symplectic form)

$$(\star) \quad \boxed{X(\omega(Y,Z)) = \mathcal{L}_X(\omega(Y,Z)) = \underbrace{(\mathcal{L}_X \omega)}_{=0}(Y,Z) + \omega([X,Y],Z) + \omega(Y,[X,Z])}$$

Then compute

$$\{f,\{g,h\}\} = \omega(X_f, X_{\{g,h\}}) = \omega(X_f, [X_h, X_g]) = -\omega(X_f, [X_g, X_h])$$

Then Jacobiator $\equiv \{f,\{g,h\}\} + \text{cyclic} \overset{?}{=} 0$

From $d\omega = 0$ we get

$$0 = d\omega(X_f, X_g, X_h) = \overbrace{X_f\,\omega(X_g,X_h) - X_g\,\omega(X_f,X_h) + X_h\,\omega(X_f,X_g)}^{①}$$
$$\underbrace{-\,\omega([X_f,X_g],X_h) + \omega([X_f,X_h],X_g) - \omega([X_g,X_h],X_f)}_{②}$$

$$= (\text{by } (\star)) \quad \left.\begin{aligned} &\omega([X_f,X_g],X_h) + \omega(X_g,[X_f,X_h]) \\ &-\omega([X_g,X_f],X_h) - \omega(X_f,[X_g,X_h]) \\ &+\omega([X_h,X_f],X_g) + \omega(X_f,[X_h,X_g]) \end{aligned}\right\} ①$$
$$+ ②$$

$$= \omega(X_g,[X_f,X_h]) + \omega(X_h[X_g,X_f]) + \omega(X_f,[X_h,X_g]) \quad \text{(cyclic)}$$

$$= \text{Jacobiator}$$

$\boxed{\text{Jacobiator} = 0}$

cf. Lee Smooth manifolds 2006
or Baum Eichfeldtheorie 2013

★ Homogeneous spaces

M smooth manifold, G Lie group acting smoothly and transitively on M (given $x, y \in M$, $\exists\, g : gx = y$; g is not unique in general)

$G \times M \to M$
$(g, p) \mapsto g \cdot p$
smooth
$g_1 \cdot (g_2 \cdot p) = g_1 g_2 \cdot p$
$e \cdot p = p$
left action

$\boxed{M : \text{homogeneous } G\text{-space}}$

If $H \subset G$ is a Lie subgroup of G

$gH = \{ gh \mid h \in H \}$ = left coset of g modulo H

G/H = set of left cosets mod H (✱)

$$\boxed{\begin{aligned} \pi : G &\longrightarrow G/H \\ g &\longmapsto gH \end{aligned}}$$ natural map

$\pi : G \to G/H$ is actually a principal bundle with structure group H

$g_1 \in gH$ iff $g_1 = gh$ for some $h \in H$

i.e. $g^{-1} g_1 = h$

$g^{-1} g_1 \in H$

★★ Theorem: G Lie group
$H \subset G$ closed Lie subgroup
✝ it is enough to assume H closed: H is automatically Lie subgroup

Then G/H has a unique smooth manifold structure
s.t. $\pi : G \to G/H$ is a smooth submersion
(π surjective, π_* surjective)

Equipped with the left action
$$g_1 \cdot (g_2 H) := (g_1 g_2) H ,$$
G/H becomes a homogeneous G-space

Example:
$SO(3) \xrightarrow{S^1} SO(3)/SO(2)$, $SO(2) \simeq U(1)$
$\underbrace{SO(3)/SO(2)}_{S^2}$, $U(1) \simeq S^1$

(✱) Remark G/H is the orbit space of the H-right action

Free action of G on M:
$g \cdot p = p \Rightarrow g = e$
(trivial isotropy)

Proof (sketch) H acts smoothly and freely on G
($gh = g \Rightarrow h = e$). The action is proper:
Let $g_i \to g$, $\{h_i\}$ such that $g_i h_i \to \gamma$
then $h_i = g_i^{-1}(g_i h_i) \to g^{-1}\gamma$. But H is closed, thus $g^{-1}\gamma \in H$
$\{h_i\}$ is convergent

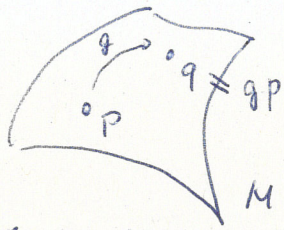
See pages III-13, III-14 for more details

Actually, homogeneous spaces are exactly of the form G/H.

The basic observation is that given a smooth G-action on M, the isotropy group $G_p = \{ g \in G \, / \, g \cdot p = p \}$ is a closed embedded Lie subgroup of G

clear more delicate (†)

If M is homogeneous, then all isotropy groups are conjugate: $\underset{\overset{\shortparallel}{q}}{G_{g \cdot p}} = g \cdot G_p \, g^{-1}$



$g' \in G_{gp}$

$$g' \circ (gp) = gp$$

$$(g'g) \cdot p = gp$$

$$\Downarrow$$

$$g^{-1} \cdot [(g'g) \cdot p] = g^{-1} \circ (gp)$$

$$(g^{-1} g' g) \cdot p = (\underbrace{g^{-1} g}_{e}) \cdot p$$

$$e \cdot p = p.$$

$$(g^{-1} g' g) \cdot p = p$$

$$\Rightarrow \quad \underbrace{g^{-1} g' g}_{\eta} \in G_p$$

$$\eta = g^{-1} g' g$$

$$\underset{G_p}{\overset{\cap}{g \eta g^{-1}}} = \underset{G_{gp}}{\overset{\cap}{g'}}$$

(†) The conclusion follows upon invoking the

★ Closed subgroup theorem

(H closed subgroup of G $\Rightarrow$ H Lie subgroup of G)

Local charts may be explicitly constructed via the Lie group theoretical Exp

★ Digression: proper actions of Lie groups on manifolds (continuous or smooth)

An action $\theta: G \times M \to M$
$(g, p) \mapsto \theta(g, p) \equiv g \cdot p$

is called **proper** if Θ $G \times M \to M \times M$
$(g, p) \mapsto (g \cdot p, p)$

Caveat: properness refers to Θ, not to the action θ

is proper (preimages of compact sets are compact)

Characterizations:

$$\boxed{① \ \Theta \text{ proper} \iff G_K = \{g \in G \,/\, (g \cdot K) \cap K \neq \emptyset\} \text{ is compact } \forall K \subset M \ K \text{ compact}}$$

Pf. $G_K = \{ g \in G \,/\, \exists p \in K \text{ with } g \cdot p \in K \}$
(clear)

$= \{ g \in G \,/\, \exists p \in M \text{ with } \Theta(g, p) \in K \times K \}$
(this means $gp \in K$, $p \in K$)

$= \{ \pi_G(\Theta^{-1}(K \times K) \}$

projection $\pi_G: G \times M \to G$
$(g, p) \to g$

$(\Rightarrow)$
If Θ is proper, $\Theta^{-1}(K \times K)$ is compact $\Rightarrow \pi_G(\Theta^{-1}(K \times K))$ is compact as well (images $f(K)$, f continuous, K compact are compact)

$(\Leftarrow)$ Let G_K be compact; then, if $L \subset M \times M$ is compact, let $K = \pi_1(L) \cup \pi_2(L)$

projections

Then $\Theta^{-1}(L) \subset \Theta^{-1}(K \times K) \subset \{ (g, p) : g \cdot p \in K, p \in K \} \subset G_K \times K$ (compact.

But $\Theta^{-1}(L)$ is closed (Θ is continuous)) hence compact [in a Hausdorff space, closed $\subset$ compact $\Rightarrow$ compact]

② Θ is proper $\Leftrightarrow$ the following holds:

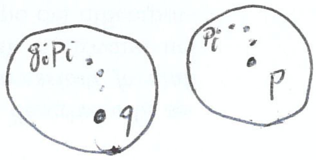
$$(*) \quad \begin{array}{l} p_i \to p \quad \text{in } M \\ \{g_i\} \text{ is such that: } g_i \cdot p_i \to q \quad \text{in } M \end{array}$$

then a subsequence of $\{g_i\}$ converges

$(\Rightarrow)$ If Θ is proper, given $\{p_i\}$ and $\{g_i\}$ satisfying $(*)$, choose $U \ni p$, $V \ni q$ precompact (compact closure)

Then p_i, $g_i \cdot p_i$ are definitely in $\bar{U} \times \bar{V}$ $\Rightarrow$ one may extract a convergent subsequence of (g_i, p_i) $\Rightarrow$ a subsequence of $\{g_i\}$ converges

$$\boxed{\Theta : (g, p) \to (g \cdot p, p)}$$



$(\Leftarrow)$ Let $(*)$ hold, and $L \subset M \times M$ be compact

Let $\{(g_i, p_i)\} \subset \Theta^{-1}(L)$, so $\Theta(g_i, p_i) = (g_i p_i, p_i) \in L$

$\Rightarrow$ passing to a subsequence, we may enforce $(*)$

Then a subsequence of $\{(g_i, p_i)\}$ converges in $G \times M$ and, since $\Theta^{-1}(L) \subset G \times M$ is closed, the limit is in $\Theta^{-1}(L)$.

Corollary: if G is a compact Lie group, any (continuous) action of G on M is proper

$(*)$ is satisfied, since any $\{g_i\} \subset G$ has a convergent subsequence

If $K = \{p\}$, then $G_K = G_p$ (isotropy group). A necessary condition for having a proper action is then G_p compact $\forall p$

TOPICS IN SYMPLECTIC AND MULTISYMPLECTIC GEOMETRY Ph.D. Course

Prof. M. Spera
UCSC - Brescia

Lecture IV

- Symplectic structure of coadjoint orbits
- Lie algebra cohomology

cf. H. Baum, Eichfeldtheorie

★ Aside: on the manifold structure of G/H
H closed Lie group of G

Basic fact: Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ — complement of $\mathfrak{h}$ (just a vector space)

$$\boxed{\begin{array}{l} \phi : \mathfrak{g} \longrightarrow G \\ \quad\;\; \underset{\mathfrak{m}}{x} + \underset{\mathfrak{h}}{y} \longmapsto \exp(x)\exp(y) \end{array}}$$

$U_{\mathfrak{m}} \times U_{\mathfrak{h}} \simeq$ is a local diffeomorphism between $\mathcal{U} \ni 0$ and $V \ni e$;

This is true in view of $\left. d\phi\right|_0 (x_i) = \frac{d}{dt} \exp(tx_i)\Big|_{t=0} = x_i$ (x or y)

One can enforce the condition $(\exp U)^{-1} \exp U \subset V$

Then consider $\varphi_{[g]} : U_{\mathfrak{m}} \xrightarrow{\exp} G \xrightarrow{L_g} G \xrightarrow{\pi} G/H$

This is a homeomorphism, whence we get a local chart

$$\left[\left(\varphi_{[g]}(U_{\mathfrak{m}}),\ \varphi_{[g]}^{-1} \right) \text{ around } [g] \in G/H \right]$$

For overlapping charts, we have — all maps involved are smooth

$$\varphi_{[a]}^{-1} \circ \varphi_{[g]} (\underset{U_{\mathfrak{m}}}{x}) = \pi_{\mathfrak{m}} \circ \phi^{-1} (L_{a^{-1}g} \exp(x))$$

$\pi_{\mathfrak{m}}$: projection $\mathfrak{g} \to \mathfrak{m}$ along $\mathfrak{h}$

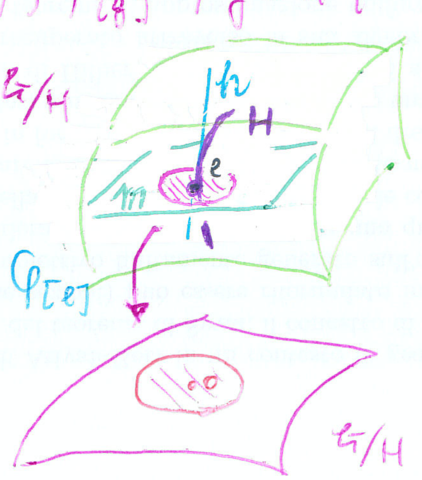
$\Rightarrow$ get $\mathcal{A} = \{ \varphi_{[g]}(\mathcal{U}_{\mathfrak{m}}),\ \varphi_{[g]}^{-1} \mid g \in G \}$

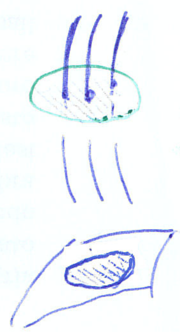
smooth atlas for G/H

$\pi : G \to G/H$ has the local representation

$$\left[\varphi_{[g]}^{-1} \circ \pi \circ (L_g \circ \phi) = \pi_{\mathfrak{m}} \right]$$

$\Rightarrow$ π is smooth





local sections for the principal bundle

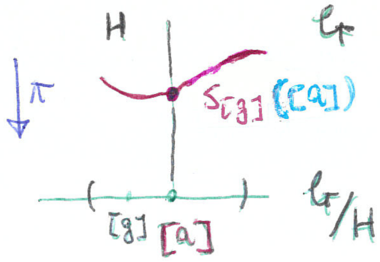
☆ Local slices for $\pi : G \to G/H$

Let $[g] \in G/H$; consider $U([g]) := \varphi_{[g]}(\mathcal{U}_m)$

and $\boxed{s_{[g]} := L_g \circ \exp \circ \varphi_{[g]}^{-1} : U([g]) \to G}$

If $[a] \in \mathcal{U}([g])$, $\exists !\ X_m \in \mathcal{U}_m$ s.t.

$$\boxed{[a]} = \varphi_{[g]}(X_m) = \pi \cdot L_g \cdot \exp X_m$$
$$= \pi \cdot L_g \cdot \exp \varphi_{[g]}^{-1}[a]$$
$$= \boxed{\pi\left(s_{[g]}([a])\right)}$$



★ On fundamental vector fields

[See e.g. H. Baum Eichfeldtheorie]

(left) Action of G on M

$X \in \mathfrak{g}$, $X^\#$ fundamental vector field pertaining to X

$$X^\#(x) := \frac{d}{dt}\Big|_{t=0} \left(\exp(-tX)\cdot x\right)$$

← in A. Cannas Da Silva +

(right action: $\frac{d}{dt}\Big|_{t=0} \left(x\cdot \exp(tX)\right)$)

Let us check that

(★) $\boxed{[X,Y]^\# = [X^\#, Y^\#]}$

(− in A. Cannas Da Silva)

also: $dr_a(X^\#) = (\mathrm{Ad}(a^{-1})X)^\#$ right action

$dl_a(X^\#) = (\mathrm{Ad}(a)X)^\#$ left action

Proof (for right actions) $x \in M$ fixed.

$$\varphi_x : G \to M, \quad g \mapsto x\cdot g$$

Let $X \in \mathfrak{g}$ (left invariant)

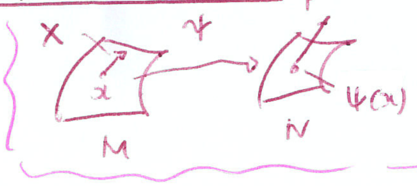
velocity of the curve $t \mapsto x\cdot a\cdot \exp(tX)$ at $t=0$

$$[(\varphi_x)_*]_a (X(a)) = (\varphi_x)_* (L_a)_* X(e) = \frac{d}{dt}\left[\varphi_x(L_a(\exp tX))\right]_{t=0}$$

$$= \frac{d}{dt}\left[x\cdot a\cdot \exp(tX)\right]\Big|_{t=0} = X^\#(x\cdot a) = X^\#(\varphi_x(a))$$

★ This calculation shows that $X \in \mathfrak{X}(G)$ and $X^\# \in \mathfrak{X}(M)$ are φ_x-related

$\left((\psi_*|_x X_x = Y_{\psi(x)} \right)$



X, Y ψ-related

whence (★) follows

The KKS symplectic structure on coadjoint orbits of Lie groups

Kirillov - Kostant - Souriau

G : Lie group $\quad \mathfrak{g}$: Lie algebra $\quad \mathfrak{g}^*$: dual of $\mathfrak{g}$

- G acts on $\mathfrak{g}$ via the adjoint action Ad

infinitesimalising to

$$ad(u) := [u, v] \qquad u, v \in \mathfrak{g}$$

(rapid check $\quad g = 1 + \varepsilon u \quad g^{-1} = 1 - \varepsilon u \ldots$)

$Ad(g)X = gXg^{-1}$, $g \in G$, $X \in \mathfrak{g}$

for matrix groups

$$[Ad(g)X]^{\#} = (R_{g^{-1}})_* X^{\#}$$

$X^{\#}$ left invariant vector field corresponding to X

$$X^{\#}(g) = (L_g)_* X \qquad X \in T_e G$$

- G acts on $\mathfrak{g}^*$ via the coadjoint action

$$\langle Ad^*(g) f, v \rangle := \langle f, Ad(g^{-1}) v \rangle \qquad g \in G,\ f \in \mathfrak{g}^*,\ v \in \mathfrak{g}$$

infinitesimalising to

$$\langle ad^*(u) f, v \rangle := - \langle f, \underbrace{[u, v]}_{ad(u)v} \rangle$$

* Coadjoint orbit $\qquad \mathcal{O}_{f_0} \cong G / G_{f_0} \quad \leftarrow$ isotropy group

closed (hence Lie) subgroup of G

$\mathcal{O}_{f_0} \subset \mathfrak{g}^*$

KKS form B

$$\boxed{B_f \big(\underbrace{ad^*(u) f}_{\equiv\, u_f^{\#}} , \underbrace{ad^*(v) f}_{\equiv\, v_f^{\#}} \big) := \langle f, [u, v] \rangle}$$

$\langle , \rangle$ duality pairing

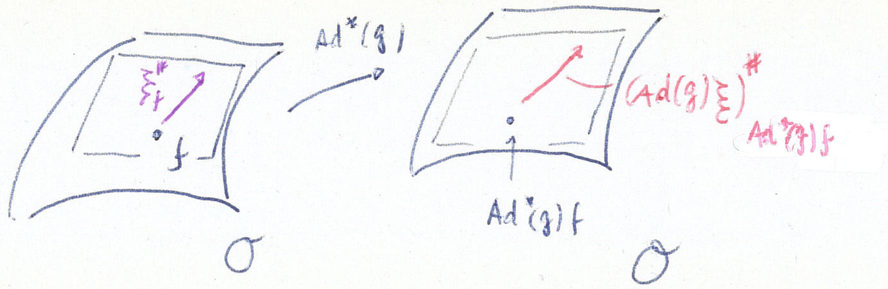
fundamental v. fields associated to $u, v \in \mathfrak{g}$

Let us check that

1. B is Ad^*-invariant
2. closed
3. non degenerate

$\Rightarrow$ B is a symplectic form on $\mathcal{O}_{f_0} \subset \mathfrak{g}^*$

① Ad^*-invariance of B



$$\boxed{B_{Ad^*(g)f}\left((Ad(g)\xi)^\#_{Ad^*(g)f},\ (Ad(g)\eta)^\#_{Ad^*(g)f}\right)} =$$

$$\langle Ad^*(g)f,\ [Ad(g)\xi,\ Ad(g)\eta]\rangle = \langle Ad^*(g)f,\ Ad(g)[\xi,\eta]\rangle$$

$$= \langle f,\ \underbrace{Ad(g^{-1})\,Ad(g)}_{\substack{Ad(g^{-1}\cdot g) \\ \| \\ Ad(e)\, =\, I}}[\xi,\eta]\rangle = \langle f, [\xi,\eta]\rangle$$

$$= \boxed{B_f(\xi_f^\#, \eta_f^\#)}$$

② Closure of B

Let B be any G-invariant 2-form on $\mathfrak{g}^*$ (or $\mathfrak{g}$)

Then

$$\boxed{dB(X,Y,Z) = B([X,Y],Z) + \text{cyclic permutations}}$$

fundamental vector fields

Indeed, if B is G-invariant we have, $\forall X, Y, Z$ f. vect. fields

$$\underbrace{X\, B(Y,Z)}_{\text{treat all these objects as arguments}} = \underbrace{(\mathcal{L}_X B)}_{=0}(Y,Z) + B(\overset{\mathcal{L}_X Y}{[X,Y]}, Z) + B(Y, \overset{\mathcal{L}_X Z}{[X,Z]})$$

From the general formula

MM: cancel out

$$dB(X,Y,Z) = \overset{B([X,Y],Z) + B(Y,[X,Z])}{X\cdot B(Y,Z)} + Y\cdot B(Z,X) + Z\cdot B(X,Y)$$

$$- \underbrace{B([X,Y],Z)}_{MM} - B([Y,Z],X) - B([Z,X],Y)$$

$$= \dots = \quad B(Y,[X,Z]) + \text{cyclic perm}$$

Now, if $B = KKS$, then (it is enough to check it on fundamental vector fields)

$$(dB)(\xi^\#_f, \eta^\#_f, \zeta^\#_f) = \langle f, \underbrace{[\xi[\eta,\zeta]] + \text{cyclic perm}}_{=0 \text{ by the Jacobi identity}} \rangle$$

(use $[\xi,\eta]^\#_f = \pm[\xi^\#_f, \eta^\#_f]$) Bornem + Cannas Da Silva −

So $\boxed{dB = 0}$

$B_f(\xi^\#_f, \zeta^\#_f) = 0 \quad \forall \zeta \in \mathfrak{g}$

③ KKS is non degenerate: $\Downarrow$

$$\langle f, [\xi, \zeta]\rangle = 0 \quad \forall \zeta$$

$$\Rightarrow \langle f, ad(\xi)\zeta\rangle = 0 \quad \forall \zeta \qquad \text{i.e.} \qquad -\langle ad^*(\xi)f, \zeta\rangle = 0$$

$$= ad^*(\xi)f = 0 \quad \Rightarrow \quad \xi \in \underset{\text{isotropy algebra}}{\mathfrak{g}_f} \quad \text{and} \quad \xi^\#_f = 0$$

★ Digression: Lie algebra cohomology

$\mathfrak{g}$ Lie algebra

$(k=0 : \Lambda^0 \mathfrak{g}^* = \mathbb{R})$
$\quad\quad\quad\quad \overset{|||}{C^0}$

$C^k := \Lambda^k \mathfrak{g}^* \equiv$ k-cochains on $\mathfrak{g}$

alternating k-linear maps $\underbrace{\mathfrak{g} \times \cdots \mathfrak{g}}_{k} \to \mathbb{R}$

Define $\delta : C^k \to C^{k+1}$ via

omission

$$\delta c\,(x_0 - x_k) = \sum_{i<j} (-1)^{i+j} c([x_i, x_j], x_0 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_k)$$

compare with the formula for $d\omega$: the terms $x_i\,\omega(x_0 \ldots \hat{x}_i \ldots)$ are missing but this is intuitively clear ... see below

Then $\delta^2 = 0$ (... Jacobi)

Let us check the formula for $k=0$ $\quad (\Lambda^0 \mathfrak{g} = \mathbb{R})$

$\delta c\,(x_0) = 0 \qquad \delta(\delta c) = 0$

$k = 1$

$$(\delta c)(x_0, x_1) = -c([x_0, x_1])$$

$$\delta(\delta c)(x_0, x_1, x_2) = -\sum_{i<j} (-1)^{i+j} \delta c([x_i, x_j], x_k) \qquad k \neq i,\ k \neq j$$

$$= -\{(-1)^{0+1} \delta c([x_0, x_1], x_2) + (-1)^{0+2} \delta c([x_0, x_2], x_1)$$
$$+ (-1)^{1+2} \delta c([x_1, x_2], x_0)\} =$$

$$= -\{-\delta c([x_0, x_1], x_2) + \delta c([x_0, x_2], x_1)$$
$$- \delta c([x_1, x_2], x_0)\}$$

$$= -\{-c([[x_0, x_1], x_2]) + c([[x_0, x_2], x_1])$$
$$- c([[x_1, x_2], x_0])\}$$

$$= +[c([[x_0, x_1], x_2] + c[[x_2\, x_0], x_1] + c([x_1, x_2], x_0])$$

$$= + c([[x_0, x_1], x_2] + \text{cyclic}) = 0$$

Form the ★ Chevalley complex

$$0 \to C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \cdots$$

$\delta^2 = 0$ implies $\mathrm{Im}\,\delta \subseteq \ker \delta$

$$\boxed{H^k(\mathfrak{g}, \mathbb{R}) := \frac{\ker \delta : C^k \to C^{k+1}}{\mathrm{Im}\,\delta : C^{k-1} \to C^k}}$$

measure obstruction to exactness

★ Chevalley cohomology groups

ker δ : cocycles
Im δ : coboundaries

If $\mathfrak{g}$ = Lie G, G compact connected Lie group, then $\boxed{H^k(\mathfrak{g}, \mathbb{R}) \cong H^k_{dR}(G, \mathbb{R})}$

Indeed, via averaging ("Weyl trick") every closed k-form on G is cohomologous to a G-invariant k-form: Sketch

$$\omega \leadsto \int_G g^*\omega \; dg =: \hat{\omega}$$

closed $d\omega = 0$

Haar measure $\int dg = 1$

$\hat{\omega}$ is G-invariant

$g^*\omega$ is closed

$d(g^*\omega) = g^* d\omega = 0$

Clearly, on G-invariant forms, the de Rham differential d reduces to δ

$$g^*\omega - \omega = \int_0^1 (\mathcal{L}_X \omega)\,dt = \int_0^1 (d\, i_X\omega + i_X \overset{0}{d\omega})\,dt$$

Cartan

$$= \int_0^1 (d\, i_X\omega)\,dt$$

$\|$ (+)

(uniform limits)

$$d \int_0^1 (i_X\omega)\,dt$$

$g = \exp X$

$g_t = \exp tX$, $t \in [0,1]$

(integrating over the compact space G)

$$\boxed{[\hat{\omega}]_{dR} = [\omega]_{dR}} \iff \boxed{\hat{\omega} = \omega + d\alpha}$$ for a suitable α

(+)
$f_i \rightrightarrows f$ K compact
$\Rightarrow \int_K f_i \to \int_K f$

$f_i \to f$
$f_i' \rightrightarrows g$
$\Rightarrow \exists f'$ and $f' = g$

★ Haar measure on $\boxed{SU(2) \cong S^3}$: standard measure on S^3 (induced from the Euclidean metric on $\mathbb{R}^4$)

descends to $\boxed{SO(3) \cong SU(2)/\mathbb{Z}_2}$
$\cong \mathbb{R}\mathbb{P}^3 \cong S^3/\sim$ ← identification of antipodal pts.

$\mathbb{Z}_2 = \pi_1(SO(3))$

$H^1(\mathfrak{g})$ and $H^2(\mathfrak{g})$

TOPICS IN SYMPLECTIC AND MULTISYMPLECTIC GEOMETRY
Ph.D. Course
Prof. M. Spera, UCSC Brescia

Lecture V

- Lie algebra cohomology (continued)
- Moment maps
- Symplectic reduction

$$c \in \mathfrak{g}^* \equiv C^1 \qquad \delta c(X_0, X_1) = -c([X_0, X_1])$$

$$[\mathfrak{g}, \mathfrak{g}] := \text{span of } [X,Y],\ X, Y \in \mathfrak{g}$$
$$= \{\text{linear combinations of } [X,Y] \,/\, X, Y \in \mathfrak{g}\}$$

commutator ideal of $\mathfrak{g}$

$$\delta c = 0 \iff c\big|_{[\mathfrak{g},\mathfrak{g}]} = 0 \qquad H^1(\mathfrak{g}, \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^0$$

no coboundary present

annihilator

$[X, Y]$, $\forall\, Y \in [\mathfrak{g}, \mathfrak{g}]$ ($X \in \mathfrak{g}$) is again an element of $[\mathfrak{g}, \mathfrak{g}]$

◇ ◇ ◇

$c \in C^2 \qquad c: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \qquad c$ skew-symmetric

$$\delta c(X_0, X_1, X_2) = -c([X_0, X_1], X_2) + c([X_0, X_2], X_1) - c([X_1, X_2], X_0)$$

$c = \delta b$, $b \in C^1$ (c coboundary) means

$$c(X_0, X_1) = \delta b(X_0, X_1) = -b([X_0, X_1])$$

☆ G semisimple: $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

"Whitehead lemmas"

$$\boxed{G \text{ semisimple} \iff H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0}$$

Example: $SO(3)$ is semisimple $\underset{\mathrm{Lie}(SO(3))}{\mathfrak{so}(3)} \cong (\mathbb{R}^3, \times)$ and any vector $v \in \mathbb{R}^3$ can be written as $v = \underline{a} \times \underline{b}$

G abelian $\Rightarrow$ not semisimple
and $G \neq \{e\}$

$U(n)$ not semisimple: it contains the torus $\underbrace{S^1 \times \ldots S^1}_{n}$ (diagonal matrices with entries $e^{i\theta_i}$)

$SU(n), SO(n)$ .. $n > 1$ are semisimple; ex: $SU(2)$ (univ. covering group of $SO(3)$) is semisimple: $SU(2) \approx S^3$ and $H^1(S^3) = H^2(S^3) = \{0\}$

From this, since cohomology can be realised by invariant forms, one would again get semisimplicity of $SU(2)$ and $SO(3)$.

G semisimple $\Leftrightarrow \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

Theorem (a) If $\boxed{H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0}$, any symplectic G-action is hamiltonian

(b) If $H^1(\mathfrak{g}) = 0$, then hamiltonian G-actions are unique

Proof Let $\psi : G \longrightarrow \mathrm{Sympl}(M, \omega)$

$$\psi_* : \mathfrak{g} \longrightarrow \mathfrak{X}^{\mathrm{sympl}}(M)$$

symplectic vector fields

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

$$\xi \longmapsto \xi^\#$$

fundamental vector field

Actually $\xi^\# \in \mathfrak{X}^{\mathrm{ham}}(M)$ ← hamiltonian vector fields

$\left([\mathfrak{X}^{\mathrm{sympl}}, \mathfrak{X}^{\mathrm{sympl}}] \subseteq \mathfrak{X}^{\mathrm{ham}} \right)$

(a) is proven provided we do not require equivariance

We look for an equivariant moment map

$$\boxed{\mu : M \longrightarrow \mathfrak{g}^*}$$

$$\boxed{\mu \circ \psi = \mathrm{Ad}^* \circ \mu}$$

symplectic action — coadjoint action

equivalently:

$\mu^* : \mathfrak{g} \longrightarrow C^\infty(M)$

comoment map

$\xi \longrightarrow \mu_\xi$

$$\mu_\xi(x) = \langle \mu(x), \xi \rangle$$

$\mu(x) \in \mathfrak{g}^*$ (moment map), $\xi \in \mathfrak{g}$

we want μ^* be a Lie algebra homomorphism

(☆) this is the infinitesimal version of equivariance

Let us check, for clarity, that the ** equivariance condition for the moment map μ at the infinitesimal level, the Lie algebra homomorphism of the comoment map μ^*

Start from

$$(\diamond) \quad \boxed{\mu(g_t \cdot x) = Ad^*(g_t)\mu} \qquad g_t = \exp t\xi$$

namely

$$\langle \mu(g_t \cdot x), \eta \rangle = \langle Ad^*(g_t)\mu, \eta \rangle \qquad \eta \in \mathfrak{g}$$

$$= \langle \mu(x), Ad(g_{-t}), \eta \rangle$$

compute $\frac{d}{dt}\Big|_{t=0}$, get

we are using Cannas Da Silva's convention

$$\langle (\xi^\# \mu)(x), \eta \rangle = -\langle \mu(x), [\xi, \eta] \rangle$$

$$= -\mu_{[\xi,\eta]}(x)$$

$//$ (∗)

$$i_{\eta^\#}\omega(\xi^\#)$$

$$= \omega(\eta^\#, \xi^\#)$$

$$= -\omega(\xi^\#, \eta^\#)$$

$$= -\{\mu_\xi, \mu_\eta\}(x)$$

recall

(∗) $i_X \omega = d\lambda_X$

$(i_X\omega)(Y) = d\lambda_X(Y)$

$= Y \cdot \lambda_X$

$$\Rightarrow (\star) \quad \boxed{\{\mu_\xi, \mu_\eta\} = \mu_{[\xi,\eta]}}$$

$(\diamond)$ is the integrated form of $(\star)$

Let us consider the following diagram:

$$\begin{tikzcd} \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r] & \mathcal{X}^{ham}(M) \\ & & \mathfrak{g} \arrow[ul, dashed] \arrow[u, "\psi_*"'] \end{tikzcd}$$

<u>Look for</u> $\mu^* \longrightarrow$ making the diagram commute

<u>and</u> yielding a Lie algebra homomorphism

Let $\forall\, \xi \in \mathfrak{g}$, $\quad \tau(\xi) = \tau_\xi \quad$ with $\quad i_{\xi^\#}\omega = d\tau_\xi$

$\xi \longmapsto \tau_\xi$ is not necessarily a Lie algebra homomorphism (τ_ξ is determined up to a constant)

But $\quad [\xi, \eta] \longmapsto \tau_{[\xi,\eta]}$

$$[\xi^\#, \eta^\#] = -[\xi,\eta]^\#$$

and $\{\tau_\xi, \tau_\eta\}$ corresponds to $-[\xi^\#, \eta^\#]$

Therefore $$\boxed{\tau_{[\xi,\eta]} - \{\tau_\xi, \tau_\eta\} \equiv c(\xi,\eta) \in \mathbb{R}}$$

$\Rightarrow$ (Jacobi) $\boxed{c \in H^2(\mathfrak{g}, \mathbb{R})}$

But, since $H^2 = 0$, $\exists\, b \in \mathfrak{g}^*$ with $\boxed{c = \delta b}$

$c(X,Y) = -b([X,Y])$. Then define

$$\boxed{\begin{aligned} \mu^* : \ & \mathfrak{g} \longrightarrow C^\infty(M) \\ & \xi \longmapsto \mu^*(\xi) = \tau_\xi + b(\xi) \equiv \mu_\xi \end{aligned}}$$

Then $\mu^*([\xi,\eta]) = \tau_{[\xi,\eta]} + b([\xi,\eta])$

$$= \{\tau_\xi, \tau_\eta\} + c([\xi,\eta])$$

$$= \{\tau_\xi, \tau_\eta\} - b([\xi,\eta]) + b([\xi,\eta])$$

$$= \{\mu_\xi, \mu_\eta\} \qquad (\mu_\xi \text{ and } \tau_\xi \text{ differ by a constant..})$$

This proves (a) i.e existence. But, actually, $H^1 = 0$ will imply uniqueness, see (b) immediately below

As for (b), given two (equivariant) moment maps μ^1 and μ^2, $\mu^1_\xi - \mu^2_\xi \equiv c(\xi)$ locally constant, i.e constant, if M is connected

so $c \in \mathfrak{g}^* : \xi \longmapsto c(\xi)$

but (μ^1 and μ^2 are Lie algebra homomorphisms)

$$c([\xi,\eta]) = 0 \implies c \in [\mathfrak{g},\mathfrak{g}]^0 = 0 \text{ if } G \text{ is semisimple}$$

in general: moment maps are unique up to elements $c \in [\mathfrak{g},\mathfrak{g}]^0$

Extreme cases

- G **semisimple** : every symplectic action is hamiltonian
- G **abelian** : symplectic actions are not hamiltonian in general; a moment map is unique up to $c \in \mathfrak{g}^*$ (in fact $[\mathfrak{g},\mathfrak{g}] = 0$, $[\mathfrak{g},\mathfrak{g}]^0 = \mathfrak{g}^*$)

Noether's Theorem (Hamiltonian setting)

Let (M, ω, H) be a hamiltonian system equipped with a G-equivariant moment map μ with G-invariant Hamiltonian H

Then μ_ξ is constant on the dynamical trajectories ("conserved current")

(+) $\omega(\xi^\#, X_H) = -\omega(X_H, \xi^\#) = -(i_{X_H}\omega)(\xi^\#) = -dH(\xi^\#) = -\xi^\#(H)$

Proof. The G-invariance of H implies

$$(*) \quad \mathcal{L}_\xi H = dH(\xi) = \xi^\#(H) = 0 \quad \forall\, \xi \in \mathfrak{g}$$

whence $\{\mu_\xi, H\} = \omega(\xi^\#, X_H) \overset{(+)}{=} -\xi^\#(H) = 0$ (X_H = ham. vector field corresponding to H)

$$\boxed{\{\mu_\xi, H\} = 0}$$

$\left(\mu^* : \mathfrak{g} \to C^\infty(M)\right.$
$\xi \mapsto \mu_\xi$

comoment map yielding a Lie algebra homomorphism

$\mu_\xi(x) = \langle \mu(x), \xi \rangle$
$\quad\; \mathfrak{g}^* \qquad \mathfrak{g}$

Thus, from Hamilton's equation

$$\boxed{\dot{\mu}_\xi = \{\mu_\xi, H\}}$$

we get $\dot{\mu}_\xi = 0$, i.e.

$$\boxed{\mu_\xi = \text{constant}}$$

(on dynamical trajectories)

★ Symplectic reduction ("elementary" approach)

f : integral of motion for a $2n$-dim. Hamiltonian system (M, ω, H)

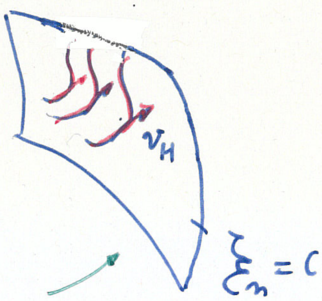
$\rightsquigarrow$ describe its trajectories in terms of a $2n-2$ dim. Hamiltonian system $(M_{red}, \omega_{red}, H_{red})$ (reduced system)

work in $\mathcal{U} \subset M$ (open), with Darboux coordinates

$$(x, \xi) \equiv (x_1 \dots x_n, \xi_1 \dots \xi_n) \qquad H = H(x, \xi)$$

let $f = \xi_n$; ξ_n first integral $\Rightarrow$

$$0 = \{\xi_n, H\} = -\frac{\partial H}{\partial x_n} \qquad \boxed{\frac{\partial H}{\partial x_n} = 0}$$



$$0 = \underbrace{\underbrace{\frac{\partial \xi_n}{\partial x_i}}_{=0} \frac{\partial H}{\partial \xi_i}}_{=0} - \underbrace{\frac{\partial \xi_n}{\partial \xi_i}}_{\delta_{ni}} \frac{\partial H}{\partial x_i} = -\delta_{ni} \frac{\partial H}{\partial x_n} = -\frac{\partial H}{\partial x_n}$$

★ trajectories stay on a level set $f = \xi_n = c$

omitted

$$\Rightarrow \quad H = H(x_1 \dots \hat{x}_n, \xi_1 \dots \xi_n)$$

Therefore, setting $\xi_n = c$, we get

$$(★) \begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial \xi_i}(\underbrace{x_1 \dots x_{n-1}}_{n-1}, \hat{x}_n, \underbrace{\xi_1 \dots \xi_{n-1}}_{n-1}, c) \\ \dot{\xi}_i = -\dfrac{\partial H}{\partial x_i}(\underbrace{x_1 \dots x_{n-1}}_{n-1}, \underbrace{\xi_1 \dots \xi_{n-1}}_{n-1}, c) \end{cases} \qquad i = 1, 2, \dots, n-1$$

together with:

$$(*) \begin{cases} \dot{x}_n = \dfrac{\partial H}{\partial \xi_n}(x_1 \dots x_{n-1}, \xi_1 \dots \xi_{n-1}, c) \\ \dot{\xi}_n = -\dfrac{\partial H}{\partial x_n} = 0 \qquad (\xi_n = c) \end{cases}$$

So we must solve (★) (a reduced Hamiltonian system) together with $(*)$, the latter yielding

$$\begin{cases} x_n(t) = x_n(0) + \displaystyle\int_0^t \frac{\partial H}{\partial \xi_n}\, dt \\ \xi_n(t) = c \end{cases}$$

★★★ Symplectic reduction

(Marsden - Weinstein ; Meyer)

(M, ω, G, μ) Hamiltonian G-space

symplectic form — ω

compact Lie group — G

(G-equivariant) moment map — μ

Let $i : \underset{\substack{\cap \\ \mathfrak{g}^*}}{\mu^{-1}(0)} \hookrightarrow M$ (inclusion)

<u>Assume</u> that G acts <u>freely</u> on $\mu^{-1}(0)$ (being G compact, the action is proper as well)

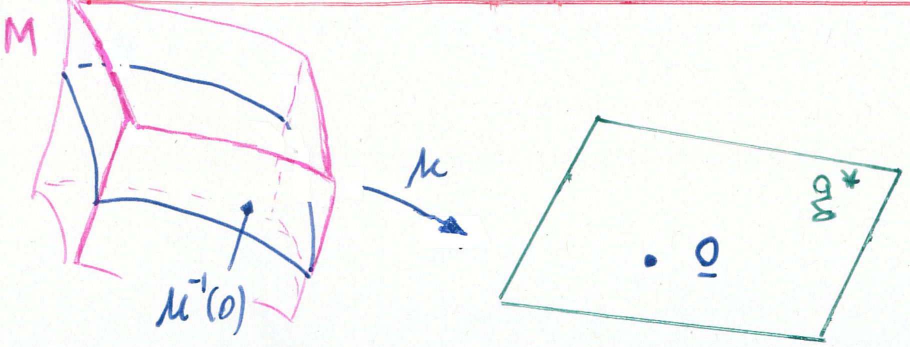
Then :

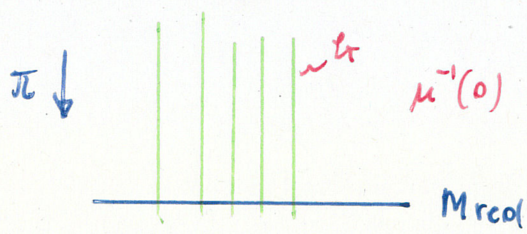
(i) $M_{red} := \mu^{-1}(0)/G$ is a smooth manifold

(ii) $\underset{\text{canonical projection}}{\pi} : \mu^{-1}(0) \longrightarrow M_{red}$ yields a principal G-bundle

(iii) $\exists$ ω_{red} on M_{red} (symplectic form) such that $i^*\omega = \pi^*\omega_{red}$

(M_{red}, ω_{red}) :	reduced space symplectic quotient MW (M) - quotient





Proof. Given $p \in M$, let $\mathfrak{g}_p = \mathrm{Lie}\, G_p$ isotropy algebra

isotropy group

Consider $\boxed{d\mu_p : T_pM \longrightarrow \mathfrak{g}^*}$

Then

$$\boxed{\begin{aligned} \mathrm{Ker}\, d\mu_p &= (T_p\mathcal{O}_p)^{\omega_p} \quad \text{symplectic orthocomplement} \\ & \qquad G\text{-orbit through } p \\ \mathrm{Im}\, d\mu_p &= \mathfrak{g}_p^0 = \{ \xi \in \mathfrak{g}^* \ / \ \langle \xi, X \rangle = 0 \ \ \forall X \in \mathfrak{g}_p \} \end{aligned}}$$

Indeed $(X \in \mathfrak{g},\ v \in T_pM)$

$\omega_p(X^\#_p, v) = \langle d\mu_p(v), X \rangle$, thus

$v \in \mathrm{Ker}\, d\mu_p \iff d\mu_p(v) = 0 \iff \omega_p(X^\#_p, v) = 0 \quad \forall X \in \mathfrak{g}$

$\iff v \in (T_p\mathcal{O}_p)^{\omega_p}$ (being $T_p\mathcal{O}_p$ generated by $X^\#_p$, $X \in \mathfrak{g}$)

Also, $X \in \mathfrak{g}_p \Rightarrow \langle d\mu_p(v), X \rangle = \omega_p(X^\#_p, v) = 0$

$\Rightarrow d\mu_p(v) \in \mathfrak{g}_p^0 \quad \forall v \in T_pM.$

Conversely $\eta \in \mathfrak{g}_p^0 \Rightarrow \langle \eta, X \rangle = 0 \quad \forall X \in \mathfrak{g}_p$

$\dim M = 2n$

$\dim G = N$

$\dim G_p = k$

$\dim \mathfrak{g}_p^0 = N - k$

$\dim \mathrm{Im}\, d\mu_p = 2n - \dim \mathrm{ker}\, d\mu_p$ (+)

$= 2n - (2n - (N-k)) = N - k$

$\dim \mathcal{O}_p = N - k$

$\quad \parallel$

$\quad G/G_p$

$\Rightarrow$ dimensions match, so $\eta = d\mu_p(v)$ for some $v \in T_pM$

(+) $T_p\mathcal{O}_p \simeq \mathfrak{g}/\mathfrak{g}_p$ $\quad \dim T_p\mathcal{O}_p = N - k$

$\dim (T_p\mathcal{O}_p)^{\omega_p} = 2n - (N-k)$

From the above we draw the following consequences:

◆ If the G-action is locally free at p
(namely G_p is discrete ($\Rightarrow \mathfrak{g}_p = 0$))

then $d\mu_p$ is surjective, i.e.

<u>p is a regular point of μ</u>

◆ If the action of G is free on $\mu^{-1}(0)$,
then $0 \in \mathfrak{g}^*$ is a regular value of μ

$\Rightarrow$ $\mu^{-1}(0)$ is a closed submanifold of M
and $\operatorname{codim}(\mu^{-1}(0)) = \dim G$ $\underset{= \dim \mathfrak{g}^*}{\overset{\dim \mathfrak{g}}{}}$

$\mu^{-1}(0)$

($\mu^{-1}(0)$ is invariant by the G-action
in view of equivariance of μ)

If $p \in \mu^{-1}(0)$, then $T_p\mu^{-1}(0) = \ker d\mu_p$ (clear...)

$T_p\mu^{-1}(0)$ and $T_p\mathcal{O}_p$ are symplectic orthocomplements

N-R

$T_p\mu^{-1}(0)$ is an isotropic subspace (of T_pM)

Indeed $\omega_{\underset{\mu^{-1}(0)}{\underset{\cap}{p}}}(X_p^\#, Y_p^\#) = \mu_{[Y,X]}(p) = 0$ $X, Y \in \mathfrak{g}$

Lecture VI

- MW reduction (continued)
- Jet spaces

** MW-reduction (continued)

We need the following

Lemma Let (V,ω) be a symplectic vector space, $I \leq V$ an isotropic subspace. Then ω induces a canonical symplectic form Ω on I^ω/I

Proof $I^\omega = \{ u \in V \mid \omega(u,w) = 0 \quad \forall w \in I \}$

Clearly $I \leq I^\omega$ (I is isotropic)

Given $u, v \in I^\omega$, let $[u], [v] \in I^\omega/I$. Set

$$\boxed{\Omega([u],[v]) := \omega(u,v)}$$

- Ω is well-defined

Indeed

$$\omega(\underset{I^\omega}{u} + \underset{I}{i}, \underset{I^\omega}{v} + \underset{I}{j}) = \omega(u,v) + \underbrace{\omega(u,j)}_{=0 \; u \in I^\omega} + \underbrace{\omega(i,v)}_{=0 \; v \in I^\omega} + \underbrace{\omega(i,j)}_{=0 \; (i,j \in I)}$$

- Ω is non degenerate

If $u \in I^\omega$ is such that

$\omega(u,v) = 0 \quad \forall v \in I^\omega$, then $u \in (I^\omega)^\omega = I$

$\Rightarrow [u] = 0$

Now recall that

If G (Lie, compact) acts on M freely (and properly, this being implied by the compactness of G), then M/G is a manifold, and $\pi: M \to M/G$ is a principal G-bundle

Therefore, we may argue as follows:

G acts freely on $\mu^{-1}(0)$

$\Rightarrow$ $d\mu_p$ is surjective $\forall p \in \mu^{-1}(0)$

$\Rightarrow$ 0 is a regular value

$\Rightarrow$ $\mu^{-1}(0)$ is a submanifold of M, of codimension $= \dim G$

This yields both (i) and (ii). Set $M_{red} = \mu^{-1}(0)/G$

we are left with checking (iii).

$T_p\mathcal{O}_p$: isotropic subspace of (T_pM, ω_p)

$(T_p\mathcal{O}_p)^{\omega} = \ker d\mu_p = T_p\,\mu^{-1}(0)$ $\quad (T_p\mathcal{O}_p)^{\omega}$ coisotropic

In view of the preceding lemma, we get a canonical symplectic structure ω_{red} on $T_p\,\mu^{-1}(0)/T_p\mathcal{O}_p$ as follows:

If $[p] \in M_{red} = \mu^{-1}(0)/G$, then

$$\boxed{T_{[p]}M_{red} \cong T_p\,\mu^{-1}(0)/T_p\mathcal{O}_p}$$

$\rightsquigarrow$ get $(\omega_{red})_p$, i.e. ω_{red} (well-defined by G-invariance)

* we must finally check that ω_{red} is closed

Clearly $\quad i^*\omega = \pi^*\omega_{red}$

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "\pi"'] & M \\ M_{red} = \mu^{-1}(0)/G & \end{tikzcd}$$

Thus $\quad \pi^* d\omega_{red} = d\pi^*\omega_{red} = d\,i^*\omega$
$= i^* d\omega = 0$

$\pi^* d\omega_{red} = 0$

But π^* is injective (π is a surjective submersion)

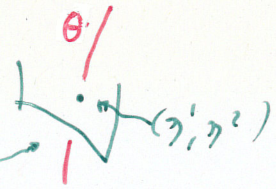
$\Rightarrow$ $\boxed{d\omega_{red} = 0}$

upshot: (M_{red}, ω_{red}) is a symplectic manifold

★ <u>Example</u>

$G = S^1 \qquad \dim M = 4$

moment map $\quad \mu : M \to \mathbb{R} \qquad p \in \mu^{-1}(0)$



(η^1, η^2) coordinates on $\mu^{-1}(0)/S^1$

θ : (angular) coordinate along $G\cdot p \equiv \mathcal{O}_p \subset M$

$$\omega = A\, d\theta \wedge d\mu + \overset{\to 0}{B_j\, d\theta\wedge d\eta^j} + C_j\, d\mu \wedge d\eta^j + D\, d\eta^1 \wedge d\eta^2$$

Now ★ $\boxed{d\mu = i_{\frac{\partial}{\partial\theta}}\,\omega} \Rightarrow A = 1 \;,\; B_j = 0 \Rightarrow$

$$\omega = d\theta \wedge d\mu + C_j\, d\mu\wedge d\eta^j + D\, d\eta^1\wedge d\eta^2 \qquad D \neq 0$$

(ω is symplectic)

$\boxed{\omega_{red} = D\, d\eta^1 \wedge d\eta^2} \rightsquigarrow i^*\omega = \pi^*\omega_{red} = D\, d\eta^1\wedge d\eta^2$

abuse of notation

Similar reasoning yields <u>MW-reduction</u> for any $f \in \mathfrak{g}^*$:

$$\boxed{M_{red} = \mu^{-1}(f) / G_f}$$

$\leftarrow$ isotropy group of $f \in \mathfrak{g}^*$

<u>Remark</u> (G-S. 1982)

X hamiltonian G-space, moment map Φ $\quad \mathcal{O}$ coadjoint orbit of $\mathfrak{g}^*$

($\mathcal{O}^-$: opposite symplectic structure). get a moment map $\Psi : X\times\mathcal{O}^- \to \mathfrak{g}^*$

$\Psi(x,f) = \Phi(x) - f \;;\; (X\times\mathcal{O}^-)_0 = \{(x,f) \,/\, \Psi(x,f) = 0\} = \{x \,/\, \Phi(x) \in \mathcal{O}\}$

set-theoretically

$X_f = \{x \in X \,/\, \Phi(x) = f\}$ $\quad$ ($X_f \equiv \mu^{-1}(f)$ in the previous notation)

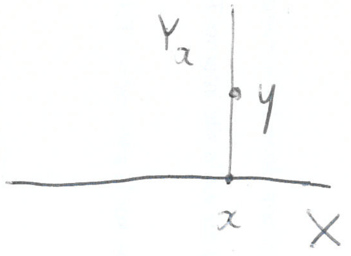
Form $\boxed{X_0 = (X\times\mathcal{O}^-)_0/G}$. Then $X_0 = X_f/G_f$

★ The first jet bundle J^1Y

(from Gotay et al 97 ; Forger-Romero 05)

$$Y \xrightarrow{\pi_{XY}} X$$

fibre bundle



$$\begin{tikzcd} J^1Y \arrow[d] \arrow[dr, dashed] & \\ Y \arrow[r] & X \end{tikzcd}$$

first jet bundle

... generalises tangent bundle

Motivation: need for an inherently finite dimensional description of classical field theory

↝ multisimplectic (n-plectic) geometry

Sticking to symplectic geometry leads to an infinite dimensional setting.

$$\boxed{(J^1Y)_y} = \{\gamma \in L(\underset{\substack{\| \\ \pi_{XY}(y)}}{T_x X}, T_yY) \,/\, T\pi_{XY} \circ \gamma = id_{T_xX}\}$$

linear maps

tangent map

↖ this is an affine space

The two pictures are reconciled via the covariant phase space, viewed multisymplectically

★ modelling vector space : $\xi = \gamma - \gamma'$ is such that

$$T\pi_{XY} \circ \xi = 0 \qquad \text{i.e.} \qquad \xi \in \ker T\pi_{XY} \equiv (VY)_y$$

vertical vectors

$\vec{J^1Y}$

see also below

$$\sim \quad \xi \in L(T_xX, V_yY) \qquad V_yY \otimes T_x^*X$$

$$\xi : \partial_\nu \longmapsto \beta^i_\nu \, \partial_i$$

★ matrix representation

fibre coordinates $y^1 \ldots y^n$

$$\gamma : \quad \partial_\nu \longmapsto \partial_\nu + \beta^i_\nu \partial_i \xmapsto{T\pi_{XY}} \partial_\nu$$

x^ν coord on X

$\partial_\nu = \delta^\mu_\nu \partial_\mu$

$$\left(\begin{array}{cccc} 1 & & 0 & \\ & 1 & & \\ 0 & & 1 & \\ & & & 1 \\ \hline \beta_1 & \beta_2 & \cdots & \end{array}\right) \qquad \xi = \left(\begin{array}{ccc} & 0 & \\ \hline \beta_1 & \beta_2 & \cdots \end{array}\right)$$

★ coordinates on J^1Y : $(x^\mu, y^i, \underbrace{y^i_\nu}_{\beta^i_\nu})$

Basic (and motivating) example

ϕ section of $Y \to X$: $\phi : X \to Y$

s.t. $\pi_{XY} \circ \phi(x) = x \quad \forall x \in X$

$(\phi(x) \in Y_x \quad \forall x)$

$\boxed{j^1\phi \equiv \text{first jet prolongation}}$

section of $J^1Y \to X$

$$\begin{tikzcd} J^1Y \arrow[d, "\pi"'] \arrow[dr, "\pi_{XY}\circ\pi"] & \\ Y \arrow[r, "\pi_{XY}"'] & X \end{tikzcd}$$

$$\boxed{j^1\phi : X \ni x \mapsto \left((x^\mu), (\phi^i(x^\mu)), \boxed{\partial_\nu \phi^i(x^\mu)} \right)}$$ "holonomic section"

"jet"

$\gamma : \quad \partial_\nu \mapsto \partial_\nu + \underbrace{\partial_\nu \phi^i}_{\beta^i_\nu} \partial_i$

of course, more general

★ This accommodates first order variations ; local equivalence of sections

$\phi \sim \phi'$ means: $\phi(x) = \phi'(x)$, $\partial_\nu \phi(x) = \partial_\nu \phi'(x)$

Basic check   coordinate transformations

Notation as in Forger-Romero CMP 2005

$$\begin{cases} x'^{\nu} = x'^{\nu}(x^{\mu}) \\ q'^{j} = q'^{j}(x^{\mu}, q^{i}) \end{cases}$$

J^1E

chain rule...

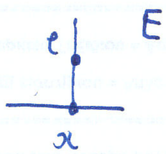
$$q'^{j}_{\nu} \equiv \frac{\partial q'^{j}}{\partial x'^{\nu}} = \underbrace{\frac{\partial q'^{j}}{\partial x^{\mu}} \cdot \frac{\partial x^{\mu}}{\partial x'^{\nu}}}_{\text{affine part}} + \underbrace{\frac{\partial q'^{j}}{\partial q^{i}} \frac{\partial q^{i}}{\partial x'^{\nu}}}_{= \frac{\partial q'^{j}}{\partial q^{i}} \cdot \frac{\partial q^{i}}{\partial x^{\mu}} \cdot \frac{\partial x^{\mu}}{\partial x'^{\nu}}}$$

$\vec{J}^1E$

skip this

$$J^1_e E = \{ \gamma \in L(T_xM, T_eE) \;/\; T_e\pi \circ \gamma = id_{T_xM} \}$$

affine subspace of $L(T_xM, T_eE)$



difference vector space:

$$\vec{J}^1_e E = \{ \vec{\gamma} \in L(T_xM, T_eE) \;/\; T_e\pi \circ \vec{\gamma} = 0 \}$$

$$= L(T_xM, V_eE) \cong T_x^*M \otimes V_eE$$

vertical part

Amplification

local coordinate transformations for J^1E

(obvious notational changes, in order to adhere to Sarlet - Romero 2005)

$$(x^\mu, q^i) \mapsto (x'^\nu, q'^i)$$

$$\begin{cases} x'^\nu = x'^\nu(x^\mu) \\ q'^i = q'^i(x^\mu, q^j) \end{cases}$$

$$\rightsquigarrow \quad (x^\mu, q^i, q^i_\nu) \mapsto (x'^\nu, q'^j, q'^j_\nu)$$

now they are independent new variables

first notice $j^1\phi(x) = T_x\varphi \in J^1_{\varphi(x)}E$

$$\partial_{\nu'}\phi' = \frac{\partial\phi'}{\partial x'^\nu} = \frac{\partial\phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\nu}$$

Recall $\boxed{\mathrm{Hom}(V, W) \cong W \otimes V^*}$ $\qquad (J^1E)_e = \text{linear maps: } T_xX \to T_eE$

$\pi(e) = x$

vector part of the transformation:

$$\boxed{q'^i_\nu = \frac{\partial q'^i}{\partial q^j} \cdot \frac{\partial x^\mu}{\partial x'^\nu} \cdot q^j_\mu}$$

But to this we must add the "translational" part coming from $q'^i = q'^i(x^\mu, q^j)$, i.e.

$$\boxed{\frac{\partial q'^i}{\partial x'^\nu} = \frac{\partial q'^i}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\nu}}$$

So, eventually:

$$\boxed{q'^i_\nu = \underbrace{\frac{\partial q'^i}{\partial q^j}\frac{\partial x^\mu}{\partial x'^\nu} q^j_\mu}_{\text{vertical part}} + \frac{\partial q'^i}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\nu}}$$

indeed displaying an affine map

Examples:

Particle mechanics

Q: configuration space $\quad X = \mathbb{R}$ (time)

$Y = \mathbb{R} \times Q \qquad \pi_{XY} : \mathbb{R} \times Q \to \mathbb{R}$

$\qquad\qquad\qquad\qquad (t, q) \mapsto t$

holonomic sections

$$\phi : X \to Y$$
$$t \mapsto (t, q)$$

curve in Q: $t \mapsto q(t)$

$$j^1\phi : X \to J^1Y$$
$$t \mapsto (t, \underbrace{q, \dot{q}}\,)$$

Clearly $\quad J^1Y = \mathbb{R} \times TQ \quad$ ⚠

affine bundle $\qquad$ vector bundle

$\dim Y_x = n$

$$J^1Y \sim (t, \underbrace{q^A, v^A}_{\text{tangent coordinates}})$$

Electromagnetism

X = 4d-spacetime $\qquad Y = \Lambda^1 X \qquad$ 4-potentials

$J^1Y \sim (x^\mu, A_\nu, v_{\nu\mu}) \qquad A = A_\nu dx^\nu$

(connections)

$$\phi : (x^\mu) \mapsto (A_\nu(x^\mu))$$

$$j^1\phi : x^\mu \longmapsto (x^\mu, A_\nu, \partial_\mu A_\nu)$$

$\boxed{\text{Abelian Chern-Simons}}$ ⋆ topological field theory

$X =$ 3-manifold $\quad Y = \Lambda^1 X \quad$ (1-forms on X)

$\phi : x \mapsto A(x)$

$J^1 Y \rightsquigarrow \quad (x^\mu, A_\nu, v_{\nu\mu})$

$j^1\phi : (x^\mu, A_\nu, \partial_\mu A_\nu)$

★ **Remark:**

A affine space

V vector space

then

$\{T : A \to V \,/\, T \text{ is affine}\}$ is a vector space

Proof. T affine: $T(\alpha^i \underset{\in A}{a_i}) = \alpha^i T a_i$ (if $\sum \alpha^i = 1$)

Einstein

Let T, S affine: then in V:

$$\boxed{(\alpha T + \beta S)(\alpha^i a_i)} = \alpha T(\alpha^i a_i) + \beta S(\alpha^i a_i)$$

$$= \alpha \alpha^i T a_i + \beta \alpha^i S a_i$$

$$= \alpha^i (\alpha T)(a_i) + \alpha^i (\beta S)(a_i)$$

$$= \alpha^i \boxed{(\alpha T + \beta S)(a_i)}$$

$\square$

★ **Dual jet bundle** $\boxed{(J^1Y)^* \longrightarrow Y}$ (a vector bundle)

generalising cotangent bundle

fibre: $(J^1Y)^*_y : \{ a : J^1_y Y \longrightarrow \Lambda^{n+1}_x X \;/\; a \text{ affine} \}$

TOPICS IN SYMPLECTIC AND MULTISYMPLECTIC GEOMETRY — Ph.D. Course

Lecture VII

Prof. M. Spera UCSC Brescia

affine space

$\Lambda^{n+1}_x X \cong \mathbb{R}$ vector space

$d^{n+1}x := dx^0 \wedge \dots \wedge dx^n$

- Dual jet bundle
- Covariant Legendre transform

$(J^1Y)^*_y$ is a vector space (see remark)

local description (p, p^μ_A) fibre coordinates

$$a : v^A_\mu \longmapsto \underbrace{(p + p^\mu_A v^A_\mu)}_{\text{affine map}}\, d^{n+1}x$$

Take $\boxed{\Lambda := \Lambda^{n+1} Y}$ Let $\boxed{Z = \{ z \in \Lambda^{n+1} Y \,/\, i_v i_w z = 0 \;\; \forall\, v, w \text{ vertical} \}}$

locally $z = p\, d^{n+1}x + p^\mu_A\, dy^A \wedge d^n x_\mu$

$d^n x_\mu \equiv \partial_\mu \lrcorner\, d^{n+1}x$

we have $\boxed{(J^1Y)^* \cong Z}$: canonical v. b. isomorphism

This is clear from the local description, but let us exhibit an explicit canonical isomorphism:

$$\Phi : Z \longrightarrow J^1Y^*$$
$$z \longmapsto \Phi(z)$$

$\gamma : T_x X \to T_y Y$

$$\boxed{\langle \Phi(z), \gamma \rangle := \gamma^* z \in \Lambda^{n+1}(X)}$$

duality

$z \in Z_y$

$\gamma \in J^1_y Y$

$x = \pi_{XY}(y)$

$\gamma \sim (v^A_\mu)$

$\gamma^* dx^\mu = dx^\mu$

$\gamma^* dy^A = v^A_\mu dx^\mu$

$$\gamma^*(\underbrace{p\, d^{n+1}x + p^{\mu}_A\, dy^A \wedge d^n x_\mu}_{\mathcal{L}})$$

$$= p\, d^{n+1}x + p^{\mu}_A v^A_\nu \underbrace{dx^\nu \wedge d^n x_\mu}_{\delta^\nu_\mu d^n x}$$

$$= (p + p^{\mu}_A v^A_\mu)\, d^{n+1}x$$

__Dual pairing__ (see Forger-Romero)

(x^μ, q^i, p^μ_i, p) $\qquad$ (x^μ, q^i, q^i_μ)

$$p_i^\mu q^i_\mu + p \qquad \text{or} \qquad (p^\mu_i q^i_\mu + p)\, d^n x$$

"untwisted" version $\qquad$ twisted version

coord. transf.

$$p'^\nu_j = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial q^i}{\partial q'^j} p^\mu_i \qquad (\text{ok})$$

trans. for p' : want

$$\boxed{p^\mu_i q^i_\mu + p = p'^\mu_i q'^i_\mu + p'}$$

This is the so-called "untwisted" (i.e. scalar) version in Forger-Romero

The twisted version (form) is worked out similarly

$$p' = p + p^\mu_i q^i_\mu - p'^\mu_i q'^i_\mu$$

inverse to each other

$$= p + p^\mu_i q^i_\mu - \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial q^i}{\partial q'^j} p^\mu_i \cdot \left[\frac{\partial q'^j}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} + \frac{\partial q'^j}{\partial q^i} \underbrace{\frac{\partial q^i}{\partial x^\mu}}_{q^i_\mu} \frac{\partial x^\mu}{\partial x'^\nu} \right]$$

$$= p - \frac{\partial q^i}{\partial q'^j} \cdot \frac{\partial q'^j}{\partial x^\mu} p^\mu_i + \underbrace{p^\mu_i q^i_\mu - p^\mu_i q^i_\mu}_{0}$$

$$\boxed{p' = p - \frac{\partial q^i}{\partial q'^j} \cdot \frac{\partial q'^j}{\partial x^\mu} p^\mu_i}$$

notice that the extra term is indeed a scalar!

★ Canonical forms

work over Z, then transfer to J^1Y^*

Θ_Λ : canonical $(n+1)$-form on Λ

$$\boxed{\Theta_\Lambda(z)(u_1 \dots u_{n+1}) = \underset{\substack{\cap \\ \Lambda^{n+1}Y}}{z}\left(T\pi_{Y\Lambda}u_1 \dots T\pi_{Y\Lambda}u_n\right)}$$

$u_1 \dots u_{n+1} \in T_z\Lambda$ $\qquad \Lambda \xrightarrow{\pi_{Y\Lambda}} Y$

$$= \boxed{(\pi^*_{Y\Lambda} z)(u_1 \dots u_{n+1})}$$

$$\boxed{\Omega_\Lambda := -d\Theta_\Lambda}$$ canonical $(n+2)$-form on Λ

$n=0$ (X 1-dim) $\Rightarrow \Lambda = T^*Y$ $\quad \Theta_\Lambda$ = canonical 1-form

$i_{\Lambda Z} : Z \to \Lambda$ inclusion

$\Theta := i^*_{\Lambda Z}\Theta_\Lambda$ canonical $(n+1)$-form on Z

$\Omega = -d\Theta = i^*_{\Lambda Z}\Omega_\Lambda$ $\qquad (d\, i^* = i^* d)$

★★★ (Z, Ω) multiphase space ("covariant phase space" caveat!)

cf. RWZ for the gauge stuff $\rightsquigarrow$ (multisymplectic manifold)

$$\boxed{z = p\, d^{n+1}x + p^\mu_A\, dy^A \wedge d^n x_\mu \equiv \Theta}$$

multimomenta

$\equiv$ abuse of language

$y^A \leftrightarrow p^\mu_A$ multimomenta

$$\boxed{\begin{aligned}\Omega = -d\Theta &= -dp \wedge d^{n+1}x - dp^\mu_A \wedge dy^A \wedge d^n x_\mu \\ &= +dy^A \wedge dp^\mu_A \wedge d^n x_\mu - dp \wedge d^{n+1}x\end{aligned}}$$

* Characterization

σ section of π_{XZ}

$$\begin{tikzcd}
\Lambda^{n+1}Y \supset Z \arrow[r, "\pi_{XZ}"] \arrow[d, "\pi_{YZ}"'] & X \arrow[l, bend left, "\sigma"] \arrow[dl, bend left, "\phi = \pi_{YZ}\circ\sigma"] \\
Y &
\end{tikzcd}$$

$$\phi = \pi_{YZ} \circ \sigma \qquad (\dagger)$$

"tautological $n+1$-form"

then
$$\boxed{\sigma^* \Theta = \phi^* \sigma}$$

$$(\sigma^* \Theta)(x)(v_1 \dots v_{n+1}) = \Theta(\sigma(x))\,(T\sigma . v_1 \text{ --- } T\sigma\, v_{n+1})$$

$\cap$
$T_x X$

$\Theta_\Lambda z = \pi_{Y\Lambda}^* z$

$$= \sigma(x)\,(\underbrace{T\pi_{YZ} \circ T\sigma}_{= T(\pi_{YZ}\circ\sigma) = T\phi}\,(v_1) \text{ ——— }) = \sigma(x)\,(T\phi\, v_1 \text{ ——— })$$

$$= (\phi^* \sigma)(x)(v_1 \text{ ——— })$$

⚠ $(\dagger)$ σ is looked upon as a form on Y (which is then pulled back by ϕ to X

* Examples

$\boxed{\text{particle mechanics}}$

$$X = \mathbb{R}\ , \quad Y = \mathbb{R} \times Q$$
$$Z = T^*Y = T^*\mathbb{R} \times T^*Q$$

coord : $(\overset{X}{\overset{\Downarrow}{t}}, \underset{\text{dual to } t}{p}, q^1 - q^N, p_1 - p_N)$ extended phase space

$$\Omega = dq^A \wedge dp_A + dt \wedge dp$$

(cf. Prologue)

$\boxed{\text{Electromagnetism}}$

$$Z \sim (x^\mu, A_\nu, p, \overset{(\text{derivative})}{y^{\nu\mu}})$$

it will become the field density

$$\Theta = y^{\nu\mu} dA_\nu \wedge d^3x_\mu + p\, d^4x$$

$$\Omega = -d\Theta = dA_\nu \wedge dy^{\nu\mu} \wedge d^3x_\mu - dp \wedge d^4x$$

valid for $g_{\mu\nu}$ treated parametrically

metric on space-time

$\boxed{\text{Chern - Simons}}$ $\quad Z : (x^\mu, A_\nu, p, p^{\nu\mu})$

$$\Theta = p^{\nu\mu} dA_\nu \wedge d^2x_\mu + p\, d^3x$$

$$\Omega = -d\Theta = dA_\nu \wedge dp^{\nu\mu} \wedge d^2x_\mu - dp \wedge d^3x$$

** The "covariant" Legendre transformation

Lagrangian density (in the orientable case, an n-form)

$$\mathcal{L} : J^1Y \to \Lambda^{n+1} X$$

$$\mathcal{L} = L(x^\mu, y^A, v^A_\mu)\, dx^{n+1}$$

covariant Legendre transform

IF stands for "fibre" ⇝

$$\boxed{\begin{gathered} \mathbb{F}\mathcal{L} : J^1Y \to J^1Y^* \cong Z \\ p_A^\mu = \frac{\partial L}{\partial v^A_\mu}, \qquad p = L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu \end{gathered}}$$

Intrinsically

$\gamma \in J^1_y Y$ $\qquad \mathbb{F}\mathcal{L}(\gamma)$ affine approx to $\mathcal{L}|_{J^1_y Y}$ at γ

$$\mathbb{F}\mathcal{L}(\gamma) : \underset{\cap\, Y_x}{J^1_y Y} \to \Lambda^{n+1}_x X$$

affine approximation to $\mathcal{L}$

$$\boxed{\langle \mathbb{F}\mathcal{L}(\gamma), \underset{\in J^1_y Y}{\gamma'} \rangle = \mathcal{L}(\gamma) + \frac{d}{d\varepsilon} \mathcal{L}(\gamma + \varepsilon(\gamma' - \gamma))\Big|_{\varepsilon = 0}}$$

$\gamma = v^A_\mu \qquad \gamma' = w^A_\mu$

recall

$$\boxed{w^A_\mu \mapsto (p + p_A^\mu w^A_\mu)\, d^{n+1}x}$$

$$\text{r.h.s.} = \left(L(\gamma) + \frac{\partial L}{\partial v^A_\mu}(w^A_\mu - v^A_\mu) \right) d^{n+1}x$$

$$\boxed{\Big\{ \underbrace{\left(L - v^A_\mu \frac{\partial L}{\partial v^A_\mu}\right)}_{p} + \underbrace{\frac{\partial L}{\partial v^A_\mu}}_{p_A^\mu} w^A_\mu \Big\}\, d^{n+1}x}$$

$\boxed{\bigstar \text{ Cartan form}}$

$$\Theta_{\mathcal{L}} := \mathbb{F}\mathcal{L}^* \Theta \qquad \Theta \text{ canonical form on } Z$$

lives on J^1Y

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}} = \mathbb{F}\mathcal{L}^* \Omega$$

Coordinate expressions

$$\boxed{\Theta_{\mathcal{L}} = \frac{\partial L}{\partial v^A_\mu} dy^A \wedge d^n x_\mu + \left(L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu\right) d^{n+1}x}$$

$$\boxed{\Omega_{\mathcal{L}} = dy^A \wedge d\left(\frac{\partial L}{\partial v^A_\mu}\right) \wedge d^n x_\mu - d\left(L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu\right) \wedge d^{n+1}x}$$

We now check that $\mathcal{L}(j^1\phi) = (j^1\phi)^* \Theta_{\mathcal{L}}$

$$\boxed{(j^1\phi)^* \Theta_{\mathcal{L}}} = \underbrace{\frac{\partial L}{\partial v^A_\mu}(j^1\phi)\, \underset{\phi^A_{,\nu} dx^\nu}{\underset{\|}{d\phi^A}} \wedge d^n x_\mu}_{} + \left(L(j^1\phi) - \underbrace{\frac{\partial L}{\partial v^A_\mu}(j^1\phi)\phi^A_{,\mu}}_{}\right) d^{n+1}x$$

← cancel out →

$$= \boxed{L(j^1\phi)\, d^{n+1}x}$$

★ Intermezzo: towards covariant moment maps

Crucial aspect of field theories:

- Liftings to jet spaces
- The E-L equations

The E-L equations both govern the evolution & impose constraints on the initial data

Dirac theory of constraints.. first class ⇝ symmetries

First task: lift automorphisms of Y to J^1Y in a way compatible, via the Legendre transform, to their canonical lifts to Z

(need a covariant analogue of the tangent map)

$\eta_Y : Y \longrightarrow Y$ π_{XY}-bundle aut. covering $\eta_X : X \to X$

$\gamma : T_x X \to T_y Y$ $\gamma \in J^1 Y$ $\boxed{\gamma : \partial_\nu \mapsto \partial_\nu + v^B_\nu \partial_B}$

$$\eta_{J^1Y}(\gamma) : T_{\eta_X(x)} X \to T_{\eta_Y(y)} Y$$

$$\begin{tikzcd} T_x X \arrow[r, "\gamma"] & T_y Y \arrow[d, "T\eta_Y"] \\ T_{\eta_X(x)} X \arrow[u, "T\eta_X^{-1}"] \arrow[r] & T_{\eta_Y(y)} Y \end{tikzcd}$$

$$\boxed{\eta_{J^1Y}(\gamma) = T\eta_Y \circ \gamma \circ T\eta_X^{-1}}$$

$\equiv J^1\eta_Y$

$\partial_B = \frac{\partial}{\partial y_B}$

$$\eta_{J^1Y}(\gamma) = \left(\eta_X^\mu(x), \eta_Y^A(x,y), \left[\partial_\nu \eta_Y^A + (\partial_B \eta_Y^A) v^B_\nu\right] \partial_\mu (\eta_X^{-1})^\nu\right)$$

v'^A_μ

(η_Y vertical $\Rightarrow \eta_X$ = identity)

compare with

$$q'^j_\nu = \frac{\partial q'^j}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\nu} + \frac{\partial q'^j}{\partial q^i} \frac{\partial q^i}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\nu}$$

v'^A_ν → replace with v^B_ν

(switch $\nu \leftrightarrow \mu$)

Let $V \in \mathfrak{X}(Y)$ $\quad \eta_\lambda$: flow of V

$$V \circ \eta_\lambda = \frac{d\eta_\lambda}{d\lambda}$$

$j^1 V := V_{J^1 Y}$: v. field on $J^1 Y$ with flow $j^1 \eta_\lambda$

$$\boxed{j^1 V \circ j^1 \eta_\lambda = \frac{d}{d\lambda} j^1 \eta_\lambda}$$

already defined

in coordinates

$$\boxed{j^1 V = \Big(\underset{\text{clear}}{V^\mu}, \underset{\text{clear}}{V^A}, \underbrace{\frac{\partial V^A}{\partial x^\mu} + \frac{\partial V^A}{\partial y^B} v^B_\mu - v^A_\nu \frac{\partial V^\nu}{\partial x^\mu}}_{\text{check this}} \Big)}$$

see next page

☆ Details :

$$\eta_X : x^\mu \longmapsto x^\mu + \varepsilon V^\mu$$

at first order...

$$\eta_X^{-1} : x^\mu \longmapsto x^\mu - \varepsilon V^\mu$$

$$\eta_Y : (x^\mu, y^A) \longmapsto (x^\mu + \varepsilon V^\mu, y^A + \varepsilon V^A)$$

$$\partial_\nu \eta_X = (\delta^\mu_\nu + \varepsilon \partial_\nu V^\mu)$$

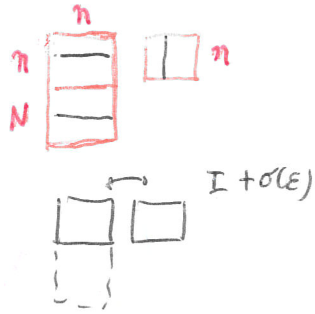
$$\partial_\nu \eta_Y = (\delta^\mu_\nu + \varepsilon \partial_\nu V^\mu , \varepsilon \partial_\nu V^A)$$

matrix representation of the **jet part**

$$\begin{array}{c|c} & \begin{array}{cc} n & N \end{array} \\ \begin{array}{c} n \\ N \end{array} & \left(\begin{array}{c|c} \delta^\mu_\nu + \varepsilon \partial_\nu V^\mu & 0 \\ \hline \varepsilon \partial_\nu V^A & \delta^A_B + \varepsilon \partial_B V^A \end{array}\right) \end{array} \begin{array}{c|c} & n \\ \begin{array}{c} n \\ N \end{array} & \left(\begin{array}{c} \delta^\mu_\nu \\ \hline v^B_\nu \end{array}\right) \end{array} \begin{array}{c|c} & n \\ n & \left(\delta^\mu_\nu - \varepsilon \partial_\nu V^\mu \right) \end{array}$$

multiply and **keep track of the ε-terms**

$$\begin{array}{c|c} & n \\ n+N & \left(\begin{array}{c} \delta^\mu_\nu + \varepsilon \partial_\nu V^\mu \\ \\ \varepsilon \partial_\nu V^A + v^A_\nu + \varepsilon \partial_B V^A v^B_\nu \end{array}\right) \end{array} \begin{array}{c|c} & n \\ n & \left(\delta^\mu_\nu - \varepsilon \partial_\nu V^\mu \right) \end{array}$$



Keep the ε-part

$$\underbrace{(n+N) \times (n+N) \cdot (n+N)}\times n$$

$$\|$$

$$(n+N) \times \underbrace{n \cdot n} \times n$$

$$n+N \times n$$

$$\partial_\nu V^A - \partial_\nu V^\mu \cdot v^A_\mu + \partial_B V^A v^B_\nu$$

$$|||$$

$$\boxed{\frac{\partial V^A}{\partial x^\mu} + \frac{\partial V^A}{\partial y^B} \cdot v^B_\mu - \frac{\partial V^\mu}{\partial x^\mu} \cdot v^A_\nu}$$

✓

Proof after GIMSY

★★★ Euler – Lagrange

$$\underbrace{\frac{\partial L}{\partial y^A}(j^1\phi) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^A_\mu}(j^1\phi)\right)}_{\frac{\delta L}{\delta \phi^A}} = 0$$

$\frac{\delta L}{\delta \phi^A}$ E-L - derivative

A posteriori E-L are intrinsic

$$Y \underset{\phi}{\overset{\pi_{XY}}{\rightleftarrows}} X$$

★ (i) ϕ stationary point of $\int_X \mathcal{L}(j^1\phi)$ (†)

(ii) E-L hold

(iii) $\forall\ W \in \mathfrak{X}(J^1Y)$,

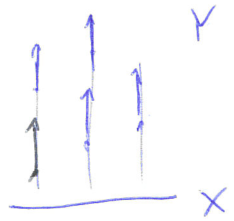
(★) $(j^1\phi)^*(W \lrcorner\, \Omega_\mathcal{L}) = 0$ $\quad \lrcorner \equiv i.$

✓ as well ok for local sections

variation by "vertical" flows

amplification $\quad L_o\ \phi_\lambda = \eta_\lambda \circ \phi \quad$ η_λ flow of V, vertical

$\lambda \in \mathbb{R}$ $\quad \eta_0 = id$ $\quad$ compactly supported on X



(†) ϕ stationary (or critical) point

$$\boxed{\frac{d}{d\lambda}\int_X \mathcal{L}(j^1\phi_\lambda)\Big|_{\lambda=0} = 0}$$

★ **Remark**

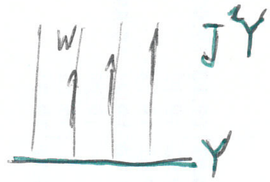
If we require (★) to hold for any section $s: X \longrightarrow J^1Y$, one has to impose regularity of $\mathbb{F}\mathcal{L}$ i.e. maximal rank or, equivalently $(J^1Y, \Omega_\mathcal{L})$ multisymplectic manifold (but this is too strong a condition)

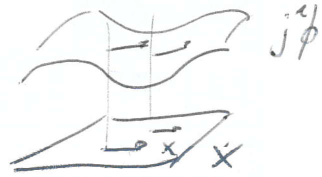
Comment: (i) $\Leftrightarrow$ (ii) is standard ...

we shall verify that (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii)

★ **Lemma** $Y \xrightarrow{\pi_{XY}} X$, ϕ ① W tangent to $\mathrm{Im}(j^1\phi)$

② or W $\pi_{Y J^1 Y}$ - vertical





then $\boxed{(j^1\phi)^*(W \lrcorner \Omega_{\mathcal{L}}) = 0}$

Proof. Assume ①, i.e. $W = T(j^1\phi)\cdot w$ for $w \in \mathfrak{X}(X)$

$$(j^1\phi)^*(T(j^1\phi)w \lrcorner \Omega_{\mathcal{L}}) =$$

$$= w \lrcorner \underbrace{\underbrace{(j^1\phi)^*\Omega_{\mathcal{L}}}_{n+2 \text{ on } X,\ (n+1)\text{-dim}}}_{\| \atop 0}$$

$f^*(i_{f_*X}\omega)$ whenever defined

$= i_X f^*\omega$

$f^*(i_{f_*X}\omega)(v_1 - v_n)$

$= (i_{f_*X}\omega)(f_*v_1 - f_*v_n)$

$= \omega(f_*X, f_*v_1 - f_*v_n)$

$= (f^*\omega)(X, v_1 - v_n)$

$= i_X(f^*\omega)(v_1 - v_n)$

$\square$

② W π_{Y, J^1Y} vertical ~ $W = (\overset{X}{0}, \overset{Y}{0}, W^A_\mu)$

$$W \lrcorner \Omega_{\mathcal{L}} = W \lrcorner \left(dy^A \wedge \overset{♦}{d\left(\frac{\partial L}{\partial v^A_\mu}\right)} \wedge d^n x_\mu - \overset{♦♦}{d\left(L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu\right)} \wedge d x^{n+1} \right)$$

keep track of dv^B_γ coefficients in

♦ $W \lrcorner \underbrace{\frac{\partial^2 L}{\partial v^A_\mu \partial v^B_\gamma} dv^B_\gamma}$ get $- \frac{\partial^2 L}{\partial v^A_\mu \partial v^B_\nu} W^B_\gamma$

second place

♦♦ $-dL + d\left(\frac{\partial L}{\partial v^A_\mu}\right) v^A_\mu + \frac{\partial L}{\partial v^A_\mu} dv^A_\mu$

As for ♦♦, in contracting with W, only this term matters $+ \frac{\partial^2 L}{\partial v^A_\mu \partial v^B_\nu} v^A_\mu dv^B_\nu$

So, finally

$$w \lrcorner \Omega_{\mathcal{L}} = -w^B_\nu \cdot \frac{\partial^2 L}{\partial v^A_\mu \partial v^B_\nu} \left[dy^A \wedge d\sigma^n_\mu - v^A_\mu \, d^{n+1}x \right]$$

pulling back via $j^1\phi$: $dy^A = \partial_\mu \phi^A dx^\mu$, $\partial_\mu\phi^A = v^A_\mu$

one gets 0

$\phi_\lambda = \eta_\lambda \circ \phi \qquad \phi_\lambda^* = \phi^* \eta_\lambda^*$

Proof of theorem. Ad (i) $\Longleftrightarrow$ (iii)

already proven

$$\frac{d}{d\lambda}\left[\int_X \mathcal{L}(j^1\phi_\lambda)\right]\Big|_{\lambda=0} = \frac{d}{d\lambda}\left[\int_X (j^1\phi_\lambda)^* \Theta_{\mathcal{L}}\right]\Big|_{\lambda=0}$$

$$= \frac{d}{d\lambda}\left[\int_X (j^1\phi)^* (j^1\eta_\lambda)^* \Theta_{\mathcal{L}}\right]\Big|_{\lambda=0}$$

Set $j^1 V := \frac{d}{d\lambda} j^1\eta_\lambda \Big|_{\lambda=0}$

$$j^1 V = \left(0,\ V^A,\ \frac{\partial V^A}{\partial x^\mu} + \frac{\partial V^A}{\partial y^B} v^B_\mu \right)$$

$\ast$ 1-jet prolongation of V

V: vertical
in general you get one extra term, see above

$\downarrow$ Lie der

$$= \int_X (j^1\phi)^* \mathcal{L}_{j^1V} \Theta_{\mathcal{L}}$$

Cartan: $-d\Theta_{\mathcal{L}}$

$-j^1V \lrcorner \Omega_{\mathcal{L}} + d(j^1V \lrcorner \Theta_{\mathcal{L}})$

$$= -\int_X (j^1\phi)^* (j^1V \lrcorner \Omega_{\mathcal{L}}) + \underbrace{\int_X d(j^1\phi)^*(j^1V \lrcorner \Theta_{\mathcal{L}})}_{=0}$$

Stokes + compact support

$$= -\int_X (j^1\phi)^* (j^1V \lrcorner \Omega_{\mathcal{L}})$$

$\Rightarrow$ (iii) $\Rightarrow$ (i) (trivial)

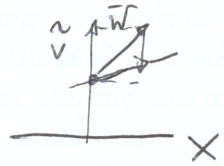
- $\mathcal{EL}$ (Continuation)
- Examples
- Covariant canonical transformations

Conversely $W \in \mathcal{X}(J^1Y)$ admits a decomposition $W = \alpha \, \mathrm{Im}\, j^1\phi + \beta \tilde{V}$

vertical

remark (ϕ defines a connection ...)

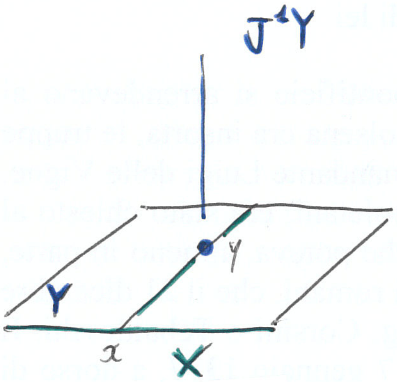
$$\begin{tikzcd} J^1Y \arrow[d] \arrow[dr] & \\ X & Y \arrow[l] \end{tikzcd}$$



J^1Y

Let $\hat{V}$ be a $\pi_{X J^1Y}$ vertical v. field. Then

$\hat{V}$ can be decomposed into a jet extension of a π_{XY}-vertical vector field and a $\pi_{Y J^1Y}$-vertical vector field



explicitly: $x^\mu \longmapsto (0, \delta y^A(x), \delta v^A_\mu(x))$

Decomposition: $x^\mu \longmapsto (0, \delta y^A(x), \tilde{v})$ ← prolongation

$x^\mu \longmapsto (0, 0, \delta v^A_\mu(x) - \tilde{v})$

So, if (i) holds, then $(*) \int_X (j^1\phi)^*(W \lrcorner \Omega_{\mathcal{L}}) = 0$

$\forall$ W on J^1Y, with compact support (use Lemma)

W can be multiplied by any scalar function $\Rightarrow$ using partitions of unity, $(*)$ holds $\forall$ v.f. $W \Rightarrow$ (f. Theor. of calculus of variations), $j^1\phi^*(W \lrcorner \Omega_{\mathcal{L}}) = 0$ $\square$

Now we prove (iii) $\Rightarrow$ (ii)

$$j^1V = \Big(0, v^A, \underbrace{\frac{\partial v^A}{\partial x^\mu} + \frac{\partial v^A}{\partial y^B} v^B_\mu}_{\tilde{\tilde{v}}}\Big)$$

Compute $(j^1\phi)^*(j^1 \underset{\pi_{XY}\text{-vertical}}{V} \lrcorner \Omega_{\mathcal{L}}) =$

$L = L(x^\mu, y^A, v^B_\mu)$

$$\left[(j^1\phi)^* \Big[\underset{(0, v^A, \tilde{v})}{j^1V} \lrcorner \Big(dy^A \wedge d\Big(\frac{\partial L}{\partial v^A_\mu}\Big) \wedge d^n x_\mu - d\Big(L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu\Big) \wedge d x^{n+1} \Big)\Big]\right]$$

significant terms: $\frac{\partial}{\partial x^\nu}\left(\frac{\partial L}{\partial v^A_\mu}\right) dx^\nu$ $\quad - \frac{\partial L}{\partial y^A} dy^A$

$$= v^A \left\{ \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^A_\mu}\right) - \frac{\partial L}{\partial y^A} \right\} (j^1\phi) \, d^{n+1}x$$

see details

Details

$$\Omega_{\mathcal{L}} = dy^A \wedge d\Big(\underbrace{\frac{\partial \mathcal{L}}{\partial v^A_\mu}}_{\chi^A_\mu}\Big) \wedge d^n x_\mu$$

$$- dL \wedge d^{n+1}x + v^A_\mu \, d\chi^A_\mu \wedge d^{n+1}x$$

$$\boxed{j^1 v = (0, v^A_i, \xi)}$$

$$+ \frac{\partial L}{\partial v^A_\mu} dv^A_\mu \wedge d^{n+1}x$$

$$j^1 v \lrcorner \Omega_{\mathcal{L}} = dy^A(j^1 v)\, d\chi^A_\mu \wedge d^n x_\mu$$

$$- \underbrace{d\chi^A_\mu (j^1 v)\, dy^A \wedge d^n x_\mu}_{\partial_\mu \phi^A \, dx^\mu} - dL(j^1 v) \wedge d^{n+1}x$$

~ cancel out

$$+ \underbrace{v^A_\mu}_{\partial_\mu \phi^A} d\chi^A_\mu (j^1 v) \wedge d^{n+1}x + \frac{\partial L}{\partial v^A_\mu} dv^A_\mu (j^1 v) \wedge d^{n+1} x$$

$$= v^A \frac{\partial}{\partial x_\mu}\Big(\frac{\partial L}{\partial v^A_\mu}\Big) d^{n+1}x - \Big(\frac{\partial L}{\partial x^\mu} dx^\mu + \boxed{\frac{\partial L}{\partial y^A} dy^A} + \frac{\partial L}{\partial v^B_\gamma} dv^B_\gamma\Big)(j^1 v)\, d^{n+1}x$$

this stays

upon contraction $\xi = 0$

this stays

cancel

cancel

$$\boxed{(j^1\phi)^*(j^1 v \lrcorner \Omega_{\mathcal{L}}) = v^A \left\{ \frac{\partial}{\partial x_\mu}\Big(\frac{\partial \mathcal{L}}{\partial v^A_\mu}\Big) - \frac{\partial \mathcal{L}}{\partial y^A} \right\} (\cdot)\, d^{n+1}x}$$

(ii) => (iii) follows from the previous arguments.

The Theorem is proved. □

$\boxed{\text{Particle mechanics}}$

$$\mathcal{L} = L(t, q, v)\, dt$$

$$p_A = \frac{\partial L}{\partial v^A} \qquad p = v^A \frac{\partial L}{\partial v^A} \quad \Xi - E$$

$$\Theta_{\mathcal{L}} = \underbrace{\frac{\partial L}{\partial v^A} dq^A}_{p_A dq^A} - E\, dt$$

$$\mathcal{E}\text{-}\mathcal{L}: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial v^A}\right) - \frac{\partial L}{\partial q^A} = 0$$

Electromagnetism

$$(x^{\mu}, A_{\nu}, p, y^{\nu\mu})$$

$g_{\mu\nu} = (+,---)$

$g \equiv \det g < 0$

$$\boxed{\Theta = y^{\nu\mu} dA_{\nu} \wedge d^3x_{\mu} + p\, d^4x}$$

$$-d\Theta = \Omega = -dy^{\nu\mu} \wedge dA_{\nu} \wedge d^3x_{\mu} - dp \wedge d^4x$$

$$= dA_{\nu} \wedge dy^{\nu\mu} \wedge d^3x_{\mu} - dp \wedge d^4x$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d^4x$$

$$F_{\mu\nu} = v_{\nu\mu} - v_{\mu\nu}$$

(ultimately $v_{\nu\mu} = \partial_{\mu} A_{\nu} \ldots$)

★ **Legendre**

(+) see box below

$$y^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\nu\mu}} = -\frac{1}{2} F^{\nu\mu} \sqrt{-g} = \frac{1}{2} F^{\mu\nu} \sqrt{-g}$$

(+)

$$\boxed{p =} \quad L - F_{\nu\mu} \frac{\partial L}{\partial F_{\nu\mu}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sqrt{-g}$$

$$- F_{\nu\mu} \left(-\frac{1}{2} F^{\nu\mu} \sqrt{-g}\right)$$

$$-\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$= \boxed{\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sqrt{-g}}$$

details of (+)

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} = -\frac{1}{4} \frac{\partial}{\partial F_{\mu\nu}} \left(F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\right) = -\frac{1}{4} F^{\mu\nu} \sqrt{-g} - \frac{1}{4} F_{\mu\nu} \frac{\partial}{\partial F_{\mu\nu}} \left(F_{\sigma\tau} g^{\mu\sigma} g^{\nu\tau}\right) \sqrt{-g}$$

$$= -\frac{1}{4} F^{\mu\nu} \sqrt{-g} - \frac{1}{4} F_{\mu\nu} \underbrace{\delta_{\sigma\mu} \delta_{\tau\nu}\, g^{\sigma\mu} g^{\tau\nu}}_{g^{\sigma\mu} g^{\tau\nu}}) \sqrt{-g}$$

$$= -\frac{1}{4} F^{\mu\nu} \sqrt{-g} - \frac{1}{4} F^{\mu\nu} \sqrt{-g}$$

{ upon switching indices

$$= -\frac{1}{2} F^{\mu\nu} \sqrt{-g} = \frac{1}{2} F^{\nu\mu} \sqrt{-g} \quad \checkmark$$

★ Euler - Lagrange (g = Minkowski)

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial F_{\nu\mu}}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$\mathcal{L} + j^\nu A_\nu$ ← source

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = j^\nu \qquad \boxed{\partial_\mu F^{\mu\nu} = j^\nu}$$

$$\partial_\mu\left(\tfrac{1}{2} F^{\mu\nu} \overset{1}{\sqrt{-g}}\right) - 0 = \tfrac{1}{2}\,\partial_\mu F^{\mu\nu} = 0$$

"full" Maxwell

$$\boxed{\partial_\mu F^{\mu\nu} = 0}$$

★ Maxwell (in vacuo)

$dA =$

$d(A_\sigma dx^\sigma)$

$= dA_\sigma \wedge dx^\sigma$

$= \partial_\mu A_\nu \, dx^\mu \wedge dx^\nu$

$$= \tfrac{1}{2}\,(\overbrace{\partial_\mu A_\nu - \partial_\nu A_\mu}^{F_{\mu\nu}})\, dx^\mu \wedge dx^\nu$$

$= \tfrac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$= F$

$dF = d(dA) = 0$

$dF = 0$

$d\, F_{\mu\nu} \wedge dx^\mu \wedge dx^\nu$

$= (\partial_\sigma F_{\mu\nu})\, dx^\sigma \wedge dx^\mu \wedge dx^\nu$

$= 0$

Bianchi identity

"Full" Maxwell: $\partial_\mu F^{\mu\nu} = j^\nu$

$$\underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{=\,0} = \partial_\nu j^\nu$$

$$\boxed{\partial_\nu j^\nu = 0}$$

★ continuity equation

Chern - Simons

$$\mathcal{L} = \frac{1}{2}\, \varepsilon^{\mu\nu\sigma} F_{\mu\nu} A_\sigma$$

$$\mathcal{L} = F \wedge A$$

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$A = A_\sigma dx^\sigma$$

$$F \wedge A = \frac{1}{2} F_{\mu\nu} A_\sigma\, dx^\mu \wedge dx^\nu \wedge dx^\sigma$$

$$= \frac{1}{2} F_{\mu\nu} A_\sigma\, \varepsilon^{\mu\nu\sigma}\, \underset{dx^1\wedge dx^2 \wedge dx^3}{\overset{\shortparallel}{d^3x}}$$

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} = \frac{1}{2}\, \varepsilon^{\mu\nu\sigma} F_{\mu\nu}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial F_{\sigma\mu}} \right) = \partial_\mu \left(\frac{1}{2}\, \varepsilon^{\sigma\mu\tau} A_\tau \right) \quad \frac{1}{2}\, \varepsilon^{\sigma\mu\tau} \partial_\mu A_\tau$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial F_{\sigma\mu}} \right) - \frac{\partial \mathcal{L}}{\partial A_\sigma} = \frac{1}{2} \left\{ \varepsilon^{\sigma\mu\tau} \partial_\mu A_\tau - \varepsilon^{\mu\nu\sigma} F_{\mu\nu} \right\} = 0$$

$$\varepsilon^{\sigma\mu\nu} \left\{ \partial_\mu A_\nu - F_{\mu\nu} \right\} = 0$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\varepsilon^{\sigma\mu\nu}\, \partial_\nu A_\mu = 0$$

$$\Leftrightarrow \quad \partial_\nu A_\mu - \partial_\mu A_\nu = F_{\nu\mu} \equiv 0$$

$F = 0$

(flat connection)

$$Z \ni (x^\mu, A_\nu, P, P^{\nu\mu})$$

$$\Theta = p^{\nu\mu}\, dA_\nu \wedge d^2x_\mu + p\, d^3x$$

$$-\Omega = dA_\nu \wedge dp^{\nu\mu} \wedge d^2x_\mu + dp \wedge d^3x$$

$$\overset{\shortparallel}{d\Theta}$$

★ Covariant canonical transformation

$\eta_Z : Z \to Z$ π_{XZ} - bundle map

covering η_X such that

$$\boxed{\eta_Z^* \Omega = \Omega}$$

$$\begin{tikzcd} Z \arrow[r, "\eta_Z"] \arrow[d, "\pi"'] & Z \arrow[d, "\pi"] \\ X \arrow[r, "\eta_X"'] & X \end{tikzcd}$$

commutative diagram

$\pi \circ \eta_Z = \eta_X \circ \pi$

★ special covariant canonical transformation

$$\boxed{\eta_Z^* \Theta = \Theta}$$

fulfils the stronger condition

Given $\eta_Y : Y \to Y$ covering η_X,

the canonical lift η_Z is defined via:

$$\boxed{\eta_Z(z) := (\eta_Y^{-1})^* z}$$

explicitly:

★ Reminder

$$Z = \left\{ z \in \Lambda^{n+1}(Y) \,/\, i_v i_w z = 0 \ \forall v, w \text{ vertical} \right\}$$

$$\eta_Z(z)(v_1 \ldots v_{n+1}) = z\left(T\eta_Y^{-1} v_1 - T\eta_Y^{-1} v_{n+1}\right)$$

$z \in Z_Y$, $v_i \in T_{\eta_Y(y)} Y$

$\eta_Y : \pi_{XY}$ - bundle map

★ notice that $T\eta_Y$ maps vertical vectors to vertical vectors

- $\eta_Z : Z \to Z$ is well-defined

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "\eta_X"'] & X \end{tikzcd}$$

Concretely:

$$z = p\, d^{n+1}x + p_A^{\mu}\, dy^A \wedge d^n x_\mu$$

Compute the various pieces...

$$(\eta_Y^{-1})^* dx^\mu = (\eta_X^{-1})^* dx^\mu$$

$$(\eta_Y^{-1})^* dy^A = \partial_\nu (\eta_Y^{-1})^A dx^\nu + \partial_B (\eta_Y^{-1})^A dy^B$$

$$P^{\mu}_{A}\, dy^{A} \wedge d\alpha^{n}_{\mu} = P^{\mu}_{A}\, dy^{A} \wedge (\partial_\mu \lrcorner\, d^{n+1}x)$$

$$(\eta^{-1}_{Y})^{*}(\partial_\mu \lrcorner\, d^{n+1}x) =$$

$$= (\eta^{-1}_{X})^{*}(\partial_\mu \lrcorner\, d^{n+1}x)$$

Now, if f is a diffeo $f: X \to X$
one has
$$f^{*}(X \lrcorner\, \omega) = (f^{-1}_{*}X) \lrcorner\, f^{*}\omega$$

Indeed $\boxed{f^{*}}(X \lrcorner\, \omega)(v_1 - v_k) = (X \lrcorner\, \omega)(f_* v_1 - f_* v_k)$

$= \omega(X, f_* v_1 - f_* v_k) = \omega(f_*(f^{-1}_{*}X), f_* v_1 - f_* v_k)$

$= (f^{*}\omega)(f^{-1}_{*}X, v_1 - v_k) =$

$= \{(f^{-1}_{*}X) \lrcorner\, f^{*}\omega\}(v_1 - v_k)$

$(\eta_*)(\partial_\mu) = \partial_\mu \eta^{\nu} \cdot \partial_\nu$ recall

$(\underbrace{\eta_* \partial_\mu}) f(y^{=\eta(x)}) =$

$\partial_\mu f(\eta(x)) =$

$\frac{\partial f}{\partial y^\nu}\frac{\partial y^\nu}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial f}{\partial y^\nu}$

$\boxed{\eta_* \partial_\mu = \frac{\partial y^\nu}{\partial x^\mu}\partial'_\nu}$

$$(\eta^{-1})^{*}(d^{n+1}x) = J^{-1} d^{n+1}x \quad (J = \mathrm{Jac}\ \eta_X)$$

$$(\eta^{-1})^{*}\, d\alpha^{n}_{\mu} = \partial_\mu \eta^{\nu} \cdot J^{-1}\, d\alpha^{n}_{\nu}$$

$$\left|\begin{array}{cc} \frac{\partial y^1}{\partial x^1} & \\ \frac{\partial y^2}{\partial x^1} & \\ \vdots & \vdots \end{array}\right|$$

differential

$y = \eta(x)$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} = \frac{\partial x}{\partial y}\frac{\partial f}{\partial x}$$

Thus

$$\boxed{\eta_Z(z) := (\eta_Y^{-1})^* z} =$$

$$p\, J^{-1} d^{n+1}x + (\eta^{-1})^* (p_A{}^{\mu}\, dy^A \wedge d^n x_\mu)$$

$$= p\, J^{-1} d^{n+1}x + p_A{}^{\mu} (\eta^{-1})^* dy^A \wedge (\eta^{-1})^* d^n x_\mu$$

$$= p\, J^{-1} d^{n+1}x + p_A^{\mu} \left[\partial_\nu (\eta^{-1})^A dx^\nu + \partial_B (\eta^{-1})^A dy^B \right]$$

$$\wedge \left[\partial_\mu \eta^\nu \cdot J^{-1} d^n x_\nu \right]$$

$$= p\, J^{-1} d^{n+1}x + \underbrace{p_A^{\mu}\, \partial_\nu (\eta^{-1})^A \partial_\mu \eta^\nu J^{-1}}_{\partial_\nu (\eta^{-1})^A p_A^{\mu} \partial_\mu \eta^\nu \cdot J^{-1}} d^{n+1}x$$

$$+ \underbrace{p_A^{\mu}\, \partial_B (\eta^{-1})^A \partial_\mu \eta^\nu \cdot J^{-1}\, dy^B \wedge d^n x_\nu}_{p^{\mu}_B\, \partial_A (\eta^{-1})^B \partial_\mu \eta^\nu J^{-1}\, dy^A \wedge d^n x_\nu}$$

$$= \left[p + \partial_\nu (\eta^{-1})^A p_A^{\mu} \partial_\mu \eta^\nu \right] \cdot J^{-1} \cdot d^{n+1}x$$

$$+ \left[p_B^{\nu}\, \partial_A (\eta_Y^{-1})^B \partial_\nu (\eta_X^{\mu}) \right] J^{-1} \cdot dy^A \wedge d^n x_\mu$$

after switching $\nu \leftrightarrow \mu$

γ

Another approach via $(J^1Y)^*$

lift of η_Y to J^1Y^*.

let

$z : J^1_y Y \to \Lambda^{n+1} X$ (an element of $(J^1_y Y)^*$)

define:

already computed

$$\langle \eta_{J^1Y^*}(z), \gamma \rangle := (\eta_X^{-1})^* \langle z, \eta^{-1}_{J^1Y}(\gamma) \rangle$$

$$z = (p, p^\mu_A) \qquad \gamma = (x^\mu, y^A, v^A{}_\mu)$$

$$\langle \eta_{J^1Y^*}(z), \gamma \rangle = (\eta_X^{-1})^* \left[(p + p^\mu_A \, \partial_\mu \eta^\nu_X [\partial_\nu(\eta_Y^{-1})^A + v^B_\nu \partial_B(\eta_Y^{-1})^A] \cdot d^{n+1}x) \right]$$

$$= \left[\underbrace{\left(p + \partial_\nu(\eta_Y^{-1})^A p^\mu_A \partial_\mu \eta^\nu_X\right) + \left(\partial_A(\eta_Y^{-1})^B p^\nu_B \partial_\nu \eta^\mu_X\right)}_{\text{comp. of } \eta_Z(z)} v^A_\mu \right] J^{-1} d^{n+1}x$$

Therefore: $\Phi : Z \to J^1Y^*$ is equivariant

with respect to η_Z & $\eta_{J^1Y^*}$

$$\boxed{\Phi \circ \eta_Z = \eta_{J^1Y^*} \circ \Phi}$$

recall:
$\Phi : Z \to J^1Y^*$
$z \mapsto \Phi(z)$
$\langle \Phi(z), \gamma \rangle := \gamma^* z$

intrinsically: $\langle \eta_{J^1Y^*}(\Phi(z)), \gamma \rangle = (\eta_X^{-1})^* \langle \Phi(z), \eta^{-1}_{J^1Y}(\gamma) \rangle$

$= (\eta_X^{-1})^* \langle \Phi(z), T\eta_Y^{-1} \circ \gamma \circ T\eta_X \rangle =$

$= (\eta_X^{-1})^* (T\eta_Y^{-1} \circ \gamma \circ T\eta_X)^* z = (\eta_X^{-1})^* (\eta_X^* \circ \gamma^* \circ \eta_Y^{-1*})(z)$

$= \gamma^* (\eta_Y^{-1})^* z = \gamma^* (\eta_Z(z)) = \langle \Phi(\eta_Z(z)), \gamma \rangle$

V_Z = lift of V one gets

$$V_Z = \Big(V^\mu,\ V^A,\ \Big|\ \underbrace{-pV^\nu_{,\nu} - p^\nu_B V^B_{,\nu}}_{①}\ \Big|\ \underbrace{p^\nu_A V^\mu_{,\nu} - p^\mu_B V^B_{,A} - p^\mu_A V^\nu_{,\nu}}_{②}\Big)$$

details: ($\delta \sim$ identity)

from $\eta_Z(z) = (\eta_Y^{-1})^* z =$

$\delta^A_\nu = 0$

$$\Big(p + \partial_\gamma \underbrace{(\eta_Y^{-1})^A}_{(\delta - \varepsilon V^A_{,\nu})}\, P_A{}^\mu\, \partial_\mu \underbrace{\eta^\nu_X}_{(\delta + \varepsilon V^\nu_{,\mu})}\Big)\, \underbrace{J^{-1}}_{(\delta - \varepsilon V^\nu_{,\nu})}\, d^{n+1}x$$

(*)

$$+ \Big(\underbrace{\partial_A (\eta_Y^{-1})^B}_{(\delta - \varepsilon V^B_{,A})}\, P_B{}^\gamma\, \underbrace{\partial_\gamma \eta^\mu_X}_{(\delta + \varepsilon V^\mu_{,\gamma})}\Big)\, J^{-1}\, dy^A \wedge d^n x_\mu$$

get: $(\eta_Y^{-1})^A = \delta - \varepsilon V^A$ $\qquad \eta^\nu_X = \delta + \varepsilon V^\nu$

$\delta^A_\gamma = 0$ (*)

$\partial_\gamma(\ ) = \delta - \varepsilon V^A_{,\gamma}$ $\qquad \partial_\gamma(\eta^\mu) = \delta + \varepsilon V^\mu_{,\gamma}$

$\partial_A(\ ) = \delta - \varepsilon V^A_{,B}$

★ keep terms of first order:

$$-\varepsilon V^\mu_{,\mu} P \quad - \varepsilon V^B_{,\nu} P^\nu_B$$

only two terms !! see (*)

$$-\varepsilon V^\nu_{,\mu} P^\mu_A - \varepsilon V^B_{,A} P^\nu_B{}^\mu + \cancel{\varepsilon P^\nu_A V^\mu_{,\nu}}$$

(*) $\boxed{\delta^A_\gamma \cdot P^\mu_A \delta^\nu_\mu = 0}$, $\delta^A_\gamma = 0$

details for M

$$\delta^B_A P^\nu_B \delta^\mu_\nu = P^\nu_A \delta^\mu_\nu = P^\mu_A$$

$$\Rightarrow P^\mu_A V^\nu_{,\nu}$$

★ covariant momentum maps

$\mathcal{G}$ Lie group $\quad \mathfrak{g} = \mathrm{Lie}(\mathcal{G})$

acting on X via diffeomorphisms and on Y, Z via bundle automorphisms

$$\eta \in \mathcal{G} \rightsquigarrow \eta_X, \eta_Y, \eta_Z$$

$$\xi \in \mathfrak{g} \rightsquigarrow \xi_X, \xi_Y, \xi_Z$$

If $\mathcal{G}$ acts on Z via covariant canonical transformations, then

$$\boxed{\mathcal{L}_{\xi_Z} \Omega = 0}$$

special canonical transformations: $\boxed{\mathcal{L}_{\xi} \Theta = 0}$

★ covariant momentum map / multimomentum map

$$\boxed{J : Z \to \mathfrak{g}^* \otimes \Lambda^n Z = \overset{\text{linear maps}}{L}(\mathfrak{g}, \Lambda^n Z)}$$

covering id on Z, such that:

$$\boxed{\underbrace{dJ(\xi)}_{n\text{-form}} = i_{\xi_Z} \overset{(n+2)\text{-form}}{\Omega}}$$

$$\boxed{J(\xi) = n\text{-form on } Z \text{ s.t. } J(\xi)_z = \underbrace{\langle \overset{\mathfrak{g}^* \otimes \Lambda^n Z}{J(z)}, \overset{\mathfrak{g}}{\xi} \rangle}_{\Lambda^n Z}}$$

Ad*-equivariance:

$$\boxed{J(\mathrm{Ad}_\eta^{-1} \xi) = \eta_Z^* [J(\xi)]}$$

$$\| $$

$$(\mathrm{Ad}^*(\eta) J)(\xi)$$

If $\mathcal{G}$ acts via special cov. canonical transformations, then

flow of $\xi \in \mathfrak{g}$ (Lee) = right translation by $\exp t\xi$

$$\boxed{\overset{n}{\widetilde{J(\xi)}} := i_{\xi_Z} \overset{n+1}{\widetilde{\Theta}}}$$

is a (special) covariant momentum map:

(cartan)

$$\boxed{dJ(\xi)} = d(i_{\xi_Z}\theta) = (\mathcal{L}_{\xi_Z} - i_{\xi_Z} d)\theta =$$

$$\underbrace{\mathcal{L}_{\xi_Z}\theta}_{=0} + i_{\xi_Z}(\overset{\Omega}{\widetilde{-d\theta}}) = \boxed{i_{\xi_Z}\Omega}$$

One has Ad^*-equivariance; check this:

$$\boxed{\eta^*}(i_\xi \overset{n}{\widetilde{\overset{n+1}{\Theta}}})(v_1 \_ v_n) = (i_\xi\theta)(\eta_* v_1 \_ \eta_* v_n) =$$

$$= \Theta(\xi, \eta_* v_1 \_ \eta_* v_n) = \theta(\eta_* \xi, \eta_* v_1 - \eta_* v_n) =$$

$$= (\eta^*\Theta)(\xi, v_1 \_ v_n) = i_\xi \underbrace{(\eta^*\theta)}_{=\Theta}(v_1 \_ v_n)$$

Special can. transf.

$$= (i_\xi \Theta)(v_1 \_ v_n) \qquad \text{But } \boxed{\xi = \eta_*^{-1}\xi = Ad_{\eta^{-1}}\xi \; (+)}$$

$$= (i_{Ad^{-1}_\eta \cdot \xi}\Theta)(v_1 \_ v_n)$$

(+) Action of η: left action $\quad \eta_* \xi = Ad_\eta \cdot \xi$

see e.g. H. Braun Each field theorie

$$\xi^\#_x = \frac{d}{dt}(\exp -t\xi \cdot x)\Big|_{t=0}$$

$$[\xi^\#, \eta^\#] = [\xi, \eta]^\#$$

Special case: action of $\mathcal{G}$ on Z ≡ lift of a $\mathcal{G}$-action on Y

Then: $J(\xi)(z) = \pi^*_{YZ}\, i_{\xi_Y} z$ is a special covariant momentum map

$$\xi_Y = T\pi_{YZ} \cdot \xi_Z$$

$$J(\xi)(z) \overset{?}{=} i_{\xi_Z} \Theta$$

Check:

$$\pi^*_{YZ}\, i_{\xi_Y} z = \pi^*_{YZ}\, i_{\underbrace{T\pi_{YZ}}_{\pi_*} \xi_Z} z$$

evaluate it on $(v_1 \ldots v_n)$

$$\pi^*_{YZ}(i_{(\ )} z)(v_1 - v_n)$$

$$= (i_{(\ )} z)(\pi_* v_1 - \pi_* v_n)$$

$$= z(\pi_* \xi_Z, \pi_* v_1 - \pi_* v_n)$$

$$= (\pi^* z)(\xi_Z, v_1 - v_n)$$

$$= i_{\xi_Z} (\pi^* z)(v_1 - v_n)$$

$$\equiv (i_{\xi_Z} \Theta)(v_1 - v_n)$$

Coordinate representation

$$\xi_Y = (\xi^\mu, \xi^A)$$

$$J(\xi)(z) = \pi^*_{YZ}\, i_{\xi_Y} z$$

$$z = p\, d^{n+1}x + p_A^{\mu}\, dy^A \wedge d^n x_\mu$$

$$\boxed{J(\xi)(z)} = p\,\xi^\mu d^n x_\mu + p_A^{\mu}\xi^A d^n x_\mu - p_A^{\mu}\xi^\nu dy^A \wedge \underbrace{d^{n-1}x_{\mu\nu}}_{\equiv\, i_{\partial_\nu} i_{\partial_\mu} d^{n+1}x}$$

Recall : momentum observables (see next page for details)

$\{f, h\} := i_{X_h} i_{X_f}\Omega$ $\qquad f(z) := \pi^*_{YZ}(V \lrcorner z)$

V: π_{XY}-projectable

presence of this term typical for multisymplectic geometry; it is absent in particle mechanics (+)

Thus $J(\xi)$: momentum observable

Prop. (*) $\boxed{\{J(\xi), J(\zeta)\} = d(i_{\xi_Z} i_{\zeta_Z}\Theta) + J([\xi,\zeta])}$

Proof. $\eta_\lambda := \exp(\lambda\zeta)$ $\qquad$ take $J(\mathrm{Ad}_\eta^{-1}\xi) = \eta_Z^*[J(\xi)]$

differentiate w.r.t. to λ, then put $\lambda = 0$, getting

$$J([\xi,\zeta]) = \mathcal{L}_{\zeta_Z} J(\xi) = d\, i_\zeta J(\xi) + i_\zeta\, dJ(\xi)$$

$$= d\, i_\zeta J(\xi) + i_\zeta i_\xi \overset{\text{“}-d\Theta\text{''}}{\Omega}$$

$$= d\, i_\zeta J(\xi) + \{J(\xi), J(\zeta)\}$$

$$d(i_\zeta i_\xi \Theta)$$

$$= -d(i_\xi i_\zeta\Theta) + \{J(\xi), J(\zeta)\}$$

$\Rightarrow$ get (*)

detail

$J(\xi) = i_\xi\Theta$

$i_\zeta\, d(i_\xi\Theta)$

$= -i_\zeta i_\xi(-\Omega)$

$= i_\zeta i_\xi \Omega$

$= \{J(\xi), J(\zeta)\}$

(+) also compare with Guillemin-Sternberg Geometric Asymptotics 1977

★ On Poisson brackets

recall (P, Ω) symplectic manifold

$$\Omega(X_f, V) = V \lrcorner df$$

Ham. v. field

$i_{X_h}\Omega = \Omega(X_h, \cdot)$

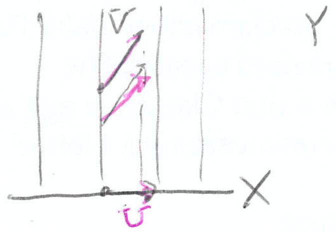
$i_{X_f}(i_{X_h}\Omega) = \Omega(X_f, X_h)$

★ Poisson brackets

$$\{f, h\} := \Omega(X_f, X_h)$$

$V \in \mathfrak{X}(Y)$ $\quad V$ π_{XY} projectable:

$$\exists\, U \in \mathfrak{X}(X) \text{ s.t. } T\pi_{XY} \circ V = U \circ \pi_{XY}$$



Now, extend the concept to the multisymplectic case

$$f(Z) := \pi^*_{YZ}(V \lrcorner Z)$$

$Z \xrightarrow{\pi_{YZ}} Y$

$f(Z)$ $\quad \underbrace{V \lrcorner Z}_{m}{}^{n+1}$

$\cap$

$\Lambda^n Z$ $\qquad f$ (n-form) the momentum observable

X_f : Hamiltonian vector field pertaining to f

$$\boxed{df = X_f \lrcorner \Omega} \qquad (\Omega \text{ non deg.})$$

PB: $\boxed{\{f, h\} := X_h \lrcorner (X_f \lrcorner \Omega)}$

Lecture XI

- Noether's theory

★ Examples of multimomentum maps

PARTICLE MECHANICS

$\mathcal{G} = \mathrm{Diff}_+(\mathbb{R})$ $X = \mathbb{R}$ $Y = \mathbb{R} \times Q$ (t, $q_1 \ldots q_N$) time reparametrizations

lifted to

$Z = T^*\mathbb{R} \times T^*Q$ $(n = 0)$ $\Lambda^1 Y = T^* Y =: Z$ (all 1-forms are vertical)

$\mathfrak{g} = \mathcal{Y}(\mathbb{R}) \equiv$ smooth functions on $\mathbb{R}$

$\mathcal{Y}(\mathbb{R})^* \equiv$ densities on $\mathbb{R}$ $\cong \mathcal{Y}_c(\mathbb{R}) \ni \chi$ ← compactly supported if we want the natural pairing to be defined (and we stick to smooth functions

otherwise one can take $\mathfrak{g} = \mathcal{Y}_c(\mathbb{R})$ and allow for distributional densities)

$\chi_Y = \chi(t) \frac{\partial}{\partial t}$; ★ momentum map:

$J : Z \longrightarrow \mathcal{Y}(\mathbb{R})^* \cong \mathcal{Y}(\mathbb{R})$

$$\boxed{\langle J(t, p, q^1 - q^N, p_1 - p_N), \chi \rangle := p\,\chi(t)}$$

time dependent transformations are encompassed

if we have a fixed time parametrization, and G acts on Q (↖ Lie group)

then $J : Z \to \mathfrak{g}^*$

$z \mapsto J(z)$ ★ usual moment map

ELECTROMAGNETISM

(fixed spacetime background)

$\mathcal{G} = \mathcal{Y}(X)$ = smooth f. on spacetime (addition)

Action: on $Y = \Lambda^1 X$, the action is the following one:

(gauge) $f \cdot A = A + df(x) \in \Lambda^1_x X$ $\chi \in \mathfrak{g} \equiv \mathcal{Y}(X)$

$\cap$ $\Lambda^1_x X$

$$\boxed{\chi_Y(A) = \chi_{,\nu} \frac{\partial}{\partial A_\nu}}$$

★ reminder $(x^\mu, A_\nu, p, \mathfrak{y}^{\nu\mu})$

$\Theta = \mathfrak{y}^{\nu\mu} dA_\mu \wedge d^3x_\mu + p\, d^4x$

$J = i_{\xi_Z} \Theta$

$$\boxed{\langle J(x, A, p, \mathfrak{y}), \chi \rangle = \mathfrak{y}^{\nu\mu} \chi_{,\nu}\, d^3x_\mu}$$

CHERN - SIMONS

$Z \ni (x^\mu, A_\nu, p, p^{\nu\mu})$

$\Theta = p^{\nu\mu} dA_\nu \wedge d^2x_\mu + p\, d^3x$

Action as before : $(\mathcal{X}_\nu(A))_\nu \frac{\partial}{\partial A_\nu}$

$$\boxed{J(x^\mu, A_\nu, p, p^{\nu\mu})(\chi) = p^{\nu\mu}\chi_{,\nu}\, d^2x_\mu}$$

CAVEAT

one can enlarge the two previous examples to encompass diffeomorphisms of X see GIMMSY

NOETHER'S theorem

Let $\mathcal{G}$ act on Y by bundle automorphisms
prolong to J^1Y via $\eta \cdot \gamma = \eta_{J^1Y}(\gamma)$

Def. $\mathcal{L}$ equivariant w.r. to $\mathcal{G}$ if $\forall \eta \in \mathcal{G}, \gamma \in J^1_xY$

$$\boxed{\mathcal{L}(\eta_{J^1Y}(\gamma)) = (\eta_x^{-1})^* \mathcal{L}(\gamma)}$$

this means: $\mathcal{L}(\gamma)$ ($n+1$ - form at x)
pushed forward to an $(n+1)$-form at $\eta_x(x)$

equality at $\eta(x)$

infinitesimally: (ξ inf. generator)

$$\boxed{\frac{\partial L}{\partial x^\mu}\xi^\mu} + \frac{\partial L}{\partial y^A}\xi^A + \frac{\partial L}{\partial v^A_\mu}\left(\xi^A_{,\mu} - v^A_\nu \xi^\nu_{,\mu} + v^B_\mu \frac{\partial \xi^A}{\partial y^B}\right)$$

$$\boxed{= -L\,\xi^\mu_{,\mu}}$$

Reminder

$\mathcal{L} : J^1Y \to \Lambda^{n+1}X$

$\mathcal{L} = L(x^\mu, y^A, v^A_\mu)\, d^{n+1}x$

$$j^1 V = \left(V^\mu, V^A, \frac{\partial V^A}{\partial x^\mu} + \frac{\partial V^A}{\partial y^B} v^B_\mu - v^A_\nu \frac{\partial V^\nu}{\partial x^\mu}\right)$$

$$\boxed{\underbrace{\frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial y^A}\xi^A + \frac{\partial L}{\partial v^A_\mu}\left(\xi^A_{,\mu} - v^A_\nu \xi^\nu_{,\mu} + v^B_\mu \frac{\partial \xi^A}{\partial y^B}\right) + L\,\xi^\mu_{,\mu}}_{\delta_\xi L} = 0}$$

$\equiv$ variation

$$\boxed{\delta_\xi L = 0}$$

Assumptions

$\boxed{\text{A1: Covariance}}$ $\mathcal{G}$ acts on Y via π_{XY}-bundle automorphisms and $\mathcal{L}$ is $\mathcal{G}$-invariant

(satisfied quite generally; exception: topological field theories)

☆ **Proposition** Let $\mathcal{L}$ be equivariant w.r.to $\mathcal{G}$ (lifted to J^1Y)

Then

(i) $\mathbb{F}\mathcal{L}$ is equivariant: $\boxed{\eta_Z \circ \mathbb{F}\mathcal{L} = \mathbb{F}\mathcal{L} \circ \eta_{J^1Y}}$

$$\begin{array}{ccc} J^1Y & \xrightarrow{\mathbb{F}\mathcal{L}} & Z \\ \downarrow{\scriptstyle \eta_{J^1Y}} & & \downarrow{\scriptstyle \eta_Z} \\ J^1Y & \xrightarrow{\mathbb{F}\mathcal{L}} & Z \end{array}$$

commutes

(ii) $\Theta_\mathcal{L}$ is invariant: $\boxed{\eta_{J^1Y}^* \Theta_\mathcal{L} = \Theta_\mathcal{L}}$ $\forall \eta \in \mathcal{G}$

(iii) $\boxed{J^\mathcal{L}(\xi) := \mathbb{F}\mathcal{L}^* J(\xi) : J^1Y \longrightarrow \Lambda^n(J^1Y)}$

is a momentum map for $\mathcal{G}$ (acting on J^1Y) w.r.to $\Omega_\mathcal{L}$:

$$\boxed{\xi_{J^1Y} \lrcorner\, \Omega_\mathcal{L} = d J^\mathcal{L}(\xi)}$$

$\xi_{J^1Y} \equiv j^1\xi_Y$ corr. to ξ

also: $\boxed{J^\mathcal{L}(\xi) = \xi_{J^1Y} \lrcorner\, \Theta_\mathcal{L}}$

Proof. Identify Z with J^1Y^*

use

- $\langle \eta_{J^1Y^*}(z), \gamma \rangle = (\eta_X^{-1})^* \langle z, \eta_{J^1Y}^{-1}(\gamma) \rangle$

&

- $\langle \mathbb{F}\mathcal{L}(\gamma), \gamma' \rangle = \mathcal{L}(\gamma) + \frac{d}{d\varepsilon} \mathcal{L}(\gamma + \varepsilon[\gamma' - \gamma])\Big|_{\varepsilon=0}$

$\star$ set $z = \mathbb{F}\mathcal{L}(\gamma)$
$(\gamma \to \gamma')$

Recall: J^1Y^*: dual jet bundle
a vector bundle fibres:
affine maps $J^1_yY \to \Lambda^{n+1}_x X$

$J^1Y^* \cong Z$ $\quad i_v i_w z = 0$ (vertical)

$z = p\, d^{n+1}x + p_A^\mu\, dy^A \wedge d x^n_\mu$

$(p, p_A^\mu) \quad v^A_\mu \longmapsto (p + p_A^\mu v^A_\mu)\, d^{n+1}x$

① $\langle \eta_{J^1Y^*}(\mathbb{F}\mathcal{L}(\gamma)), \gamma' \rangle = (\eta_X^{-1})^* \langle \mathbb{F}\mathcal{L}(\gamma), \eta^{-1}_{J^1Y}(\gamma') \rangle$

$$= (\eta_X^{-1})^* \left[\mathcal{L}(\gamma) + \frac{d}{d\varepsilon} \mathcal{L}(\gamma + \varepsilon(\cdot - \gamma))\Big|_{\varepsilon=0} \right]$$

On the other hand

② $\langle \mathbb{F}\mathcal{L}(\eta_{J^1Y}(\gamma)), \gamma' \rangle = \mathcal{L}(\eta_{J^1Y}(\gamma)) + \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{L}(\eta_{J^1Y}(\gamma) + \varepsilon(\gamma' - \eta_{J^1Y}(\gamma)))$

Thus ① = ② by

equivariance see

Equivariance:
$\mathcal{L}(\eta_{J^1Y}(\gamma)) = (\eta_X^{-1})^* \mathcal{L}(\gamma)$

Cartan form
$\Theta_{\mathcal{L}} = \mathbb{F}\mathcal{L}^* \Theta$
$= \frac{\partial L}{\partial v^A_\mu} dy^A \wedge d^n_\mu x + \left(L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu\right) d^{n+1}x$

(ii) ok at least for special actions
(see also AM)

(iii): infinitesimally one has

$$\xi_Z \circ \mathbb{F}\mathcal{L} = T\mathbb{F}\mathcal{L} \circ \xi_{J^1Y} \quad \text{(clear)}$$

via pulling back (via $\mathbb{F}\mathcal{L}$) $\boxed{dJ(\xi) = i_{\xi_Z}\Omega}$

and using the previous formula we get the sought for formulae

★ $J^{\mathcal{L}}$: covariant momentum map in the Lagrangian representation

we get

$$\boxed{J^{\mathcal{L}}(\xi) = \left(\frac{\partial L}{\partial v^A_\mu}\xi^A + \left[L - \frac{\partial L}{\partial v^A_\nu}v^A_\nu\right]\xi^\mu\right)d^n x_\mu - \frac{\partial L}{\partial v^A_\mu}\xi^\nu dy^A \wedge d x^{n-1}_{\mu\nu}}$$

if ϕ solves E-L we have $\boxed{(j^1\phi)^*(W \lrcorner \Omega_{\mathcal{L}}) = 0}$

$\forall$ W v.f. on J^1Y.

So set $W = \xi_{J^1Y}$, apply $(j^1\phi)^*$ to

$$\boxed{\xi_{J^1Y} \lrcorner \Omega_{\mathcal{L}} = d J^{\mathcal{L}}(\xi)}$$

and get

★★ **Theorem** (Noether's theorem - divergence form)
under condition A1, $\forall \xi \in \mathfrak{g}$

$$\boxed{d\Big[\underbrace{(j^1\phi)^* J^{\mathcal{L}}(\xi)}_{\text{★ Noether current}}\Big] = 0} \qquad \forall \phi \text{ sol of E.L}$$

"First Noether theorem"

Work out a coordinate expression for

$$\boxed{(j^1\phi)^* J^{\mathcal{L}}(\xi)}$$

★ Noether current !

not necessarily fulfilling E-L

$$= \left(\frac{\partial L}{\partial v^A_\mu}(j^1\phi)(\xi^A\circ\phi) + L(j^1\phi)\xi^\mu - \frac{\partial L}{\partial v^A_\nu}(j^1\phi)\,\phi^A_{,\nu}\xi^\mu \right) d^n x_\mu$$

$$- \frac{\partial L}{\partial v^A_\mu}(j^1\phi)\,\phi^A_{,\lambda}\xi^\nu \underbrace{dx^\lambda \wedge d^{n-1}x_{\mu\nu}}_{\delta^\lambda_\nu d^n x_\mu - \delta^\lambda_\mu d^n x_\nu}$$

Now use:

$$\boxed{dx^\lambda \wedge d^{n-1}x_{\mu\nu} = \delta^\lambda_\nu d^n x_\mu - \delta^\lambda_\mu d^n x_\nu}$$

Check: $dx^\lambda \wedge i_{\partial_\nu}(i_{\partial_\mu} d^{n+1}x) = \underbrace{dx^\lambda}_{\omega} \wedge i_\nu(\underbrace{d^n x_\mu}_{\phi})$ $\quad X = \partial_\nu$

Recall

$$\left(i_X(\omega\wedge\phi) = i_X\omega\wedge\phi + \underbrace{(-1)^k\omega\wedge i_X\phi}_{-\omega\wedge i_X\phi} \right) \quad k=1$$

$$\Rightarrow \quad i_\nu(dx^\lambda)\wedge d^n x_\mu - i_\nu(dx^\lambda\wedge d^n x_\mu)$$

$$= \delta^\lambda_\nu d^n x_\mu - \delta^\lambda_\mu d^n x_\nu$$

$\pm\delta^\lambda_\mu d^{n+1}x$

This is the correct sign

$i_{\partial_\nu} d^{n+1}x$

$= \pm dx^0 \wedge \widehat{dx^\nu} \wedge dx^n$

after cancellation

$$= \boxed{\left[\frac{\partial L}{\partial v^A_\mu}(j^1\phi)\underbrace{(\xi^A\circ\phi - \phi^A_{,\nu}\xi^\nu)}_{-\mathcal{L}_\xi\phi} + L(j^1\phi)\xi^\mu\right] d^n x_\mu}$$

★ This is the desired expression for any section ϕ

$$(\mathcal{L}_\xi\phi)^A = \phi^A_{,\nu}\xi^\nu - \xi^A\circ\phi$$

Lecture XII

- Noether currents (continued)
- Vertical transitivity
- Examples

In order to compute $d\,[(j^1\phi)^* J^{\xi}(\xi)]$ we need another identity:

$$\boxed{d\,(v^\mu d^n x_\mu) = \partial_\rho v^\mu dx^\rho \wedge d^n x_\mu = \partial_\mu v^\mu \, d^{n+1}x}$$

justifies "div"

Then

$$d\,[(j^1\phi)^* J^{\xi}(\xi)] = \partial_\mu \Big[\frac{\partial L}{\partial v^A_\mu}(j^1\phi)\,(\xi^A\circ\phi - \phi^A_{,\nu}\xi^\nu) + L(j^1\phi)\,\xi^\mu \Big] d^{n+1}x \qquad (\star)$$

$$= \Big\{ \partial_\mu\Big[\frac{\partial L}{\partial v^A_\mu}(j^1\phi)\Big]\,\underbrace{(\xi^A\circ\phi - \phi^A_{,\nu}\xi^\nu)}_{-[\phi^A_{,\nu}\xi^\nu - \xi^A\circ\phi]}$$

$$+ \frac{\partial L}{\partial v^A_\mu}(j^1\phi)\,\partial_\mu\big[\xi^A\circ\phi - \phi^A_{,\nu}\xi^\nu\big] \quad \text{justified}$$

$$+ \partial_\mu\big(L(j^1\phi)\big)\,\xi^\mu + L(j^1\phi)\,\xi^\mu_{,\mu} \Big] \Big\} d^{n+1}x \qquad (\star)$$

add & subtract

$$\frac{\partial L}{\partial y^A}(j^1\phi)\,(-\xi^A\circ\phi + \phi^A_{,\nu}\xi^\nu)$$

$$+ \frac{\partial L}{\partial y^A}(j^1\phi)\,\phi^A_{,\nu}\xi^\nu$$

contributes to $\delta_\xi L$

recall

$$j^1 V = \Big(V^\mu, V^A, \frac{\partial V^A}{\partial x^\mu} + \frac{\partial V^A}{\partial y^B}v^B_\mu - v^A_\nu \frac{\partial V^\nu}{\partial x^\mu}\Big)$$

$$V = \xi$$

$$\delta_\xi L = \frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial y^A}\xi^A + \frac{\partial L}{\partial v^A_\mu}\Big(\xi^A_{,\mu} - v^A_\nu\xi^\nu_{,\mu} + v^B_\mu\frac{\partial \xi^A}{\partial y^B}\Big) + L\,\xi^\mu_{,\mu}$$

$$\boxed{\frac{\delta L}{\delta\phi^A} = \frac{\partial L}{\partial y^A}(j^1\phi) - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial L}{\partial v^A_\mu}(j^1\phi)\Big)}$$

$\star$ variational derivative

claim $(\star) = \Big\{ \frac{\delta L}{\delta \phi^A}(\varepsilon_\xi\phi)^A + \delta_\xi L \Big\}(j^1\phi)\, d^{n+1}x$

check

★ compute backwards:

$$\left[\left\{ \frac{\delta L}{\delta \phi^A} (\mathcal{L}_\xi \phi)^A + \delta_\xi L \right\} (j^1\phi) \right.$$

$$= \left\{ \frac{\partial L}{\partial y^A}(j^1\phi) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^A_\mu}(j^1\phi)\right) \right\} \left\{ \phi^A_{,\nu}\xi^\nu - \xi^A \circ \phi \right\}$$

cancel out

$$+ \frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial y^A}\xi^A + \frac{\partial L}{\partial v^A_\mu}\left(\xi^A_{,\mu} - v^A_\nu \xi^\nu_{,\mu} + v^B_\mu \frac{\partial \xi^A}{\partial y^B}\right) + L\xi^\mu_{,\mu}$$

$$= -\partial_\mu\left(\frac{\partial L}{\partial v^A_\mu}\right)\left\{ \phi^A_{,\nu}\xi^\nu - \xi^A\circ\phi \right\}$$

$$+ \frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial v^A_\mu}\left(\xi^A_{,\mu} - v^A_\nu \xi^\nu_{,\mu} + v^B_\mu \frac{\partial \xi^A}{\partial y^B}\right)$$

(get ★)

upshot:

$$\boxed{d\left[(j^1\phi)^* J^L(\xi)\right] = \left\{ \frac{\delta L}{\delta\phi^A}(\mathcal{L}_\xi\phi)^A + \delta_\xi L \right\} d^{n+1}x}$$

If ϕ satisfies E-L & one has $\mathcal{G}$-invariance

$$\frac{\delta L}{\delta\phi^A} = 0, \quad \delta_\xi L = 0 \quad \Rightarrow \quad \underline{\text{NOETHER's Theorem}}$$

□

So EL + equivariance $\Rightarrow$ Noether

$\Leftarrow$

try to get a converse

$\mathcal{G}$-action vertically transitive:

$\forall x \in X$, $\forall y \in Y_x$, $\forall \phi$ local section through y

$$\{(\mathcal{L}_\xi \phi)(x) \,/\, \xi \in \mathfrak{g}\} = V_y Y \quad \text{vertical bundle}$$

(fields at x are sent to any other value)

★ **Theorem.** Assume $\mathcal{G}$-equivariance and $\mathcal{G}$ acting in a vertically transitive fashion on Y.

Then ϕ satisfies $EL \Leftrightarrow$ Noether conservation law holds $\forall \xi$

Proof: clear. $\delta_\xi L = 0$ and $\dfrac{\delta L}{\delta \phi^A} (\mathcal{L}_\xi \phi)^A = 0 \quad \forall \xi$

Therefore, by vertical transitivity one gets $\dfrac{\delta L}{\delta \phi^A} = 0$, i.e. E-L. □

★ vertical transitivity: equivalent formulations

$$\mathfrak{g}(y) := \{ \xi_Y(y) \,/\, \xi \in \mathfrak{g} \}$$

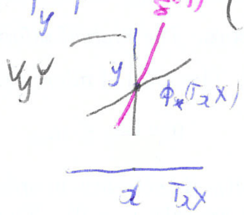
span of infin. generators

$$\equiv \quad T_y(G \cdot y)$$

vertical transitivity equivalent to either

(i) $\forall y \in Y, \ \forall \phi, \ \phi(x) = y,$

$$\mathrm{Im}\, \phi_*(T_x X) + \mathfrak{g}(y) = T_y Y \quad \text{(transversality)}$$



$x \mapsto \phi(x)$

$\phi_* : T_x X \to T_{\phi(x)} Y$

$\phi_*(T_x X)$

(ii) $\forall y \in Y, \ \forall (x^\mu, y^A),$

$$\{ \xi^A(y)\, \partial_A \,/\, \xi \in \mathfrak{g} \} = V_y Y$$

$$(\xi_Y = \xi^\mu \partial_\mu + \xi^A \partial_A)$$

⚠

(vertical transitivity does not imply verticality of ξ_Y)

Also $\quad V(\xi_Y(\phi(x))) = -(\mathcal{L}_\xi \phi)(x)$

vertical component

recall $\quad (\mathcal{L}_\xi \phi)^A = \phi^A_{,\nu}\, \xi^\nu - \xi \circ \phi$

Examples

Particle mechanics

$$\left[\frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial y^A}\xi^A + \frac{\partial L}{\partial v^A_\mu}\left(\xi^A_{,\mu} - v^A_\nu \xi^\nu_{,\mu} + v^B_\mu \frac{\partial \xi^A}{\partial y^B}\right) + L\,\xi^\mu_{,\mu} = 0 \right]$$

$\mathcal{L}_\xi L = 0$

$\mathrm{Diff}_+(\mathbb{R})$ - equivariance

(time reparametrization invariant)

$$\frac{\partial L}{\partial t} = 0 \qquad \xi = \chi(t)\partial_t$$

$$\frac{\partial L}{\partial v^A}\left(-v^A\dot{\chi}\right) + L\cdot\dot{\chi} = 0$$

$$\Big(\underbrace{L - \frac{\partial L}{\partial v^A}v^A}_{E}\Big)\dot{\chi} = 0 \qquad \forall\, \chi$$

$$E = L - \frac{\partial L}{\partial v^A}v^A = 0$$

⇝ see Prologue

Let G act on Q ⇝ prolongation leaves L invariant.

Let $\mathcal{G} = \mathrm{Diff}_+(\mathbb{R}) \times G \qquad \mathfrak{g} \ni (\chi, \xi)$

$$\left[(j^1\phi)^* J^L(\chi,\xi) = \frac{\partial L}{\partial v^A}\left(\xi^A - v^A\chi\right) + L\,\chi\right.$$

$$= \underbrace{\Big(L - \frac{\partial L}{\partial v^A}v^A\Big)}_{=0}\chi + \left.\frac{\partial L}{\partial v^A}\xi^A\right]$$

$$= \underbrace{P_A\,\xi^A}_{J^L(\xi)}$$

$$(j^1\phi)^* J^L(\xi) = \left\{ \frac{\partial L}{\partial v^A_\mu}\Big(\underbrace{\xi^A\circ\phi - \phi^A_{,\nu}\xi^\nu}_{-(\mathcal{L}_\xi \phi)^A}\Big) + L(j\phi)\,\xi^\mu \right\} d^n x_\mu$$

$\phi^A_{,\nu} \rightsquigarrow v^A_\nu$

★ Noether: $\boxed{\dfrac{d}{dt} J^L(\xi) = 0}$

vertical transitivity = transitivity of G-action on Q

For a relativistic free particle, taking length as a Lagrangian, we have a diffeomorphism invariant theory

(we have $p = 0$ and $\mu \equiv 0$)

If $(Q, g) = $ Minkowski, $G = $ Poincaré group

$\frac{d}{dt} J^{L}(\xi) = 0$ means:

energy-momentum
angular momentum are constant

along trajectories. That is: dynamical trajectories are geodesics

Electromagnetism

$$\mathcal{G} = \mathcal{Y}(x) \qquad Y = \Delta^1 X$$

Maxwell Lagrangian: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d^4x$

(invariant under prol. of the action of $\mathcal{Y}(x)$)

$$d\left[(j^1 A)\, J^{\mathcal{L}}(X)\right] = d\left[(jA)^* F^{\mu\nu} \sqrt{-g}\, X_{,\nu}\, d^3x_\mu\right]$$

$$= d\left[(A^{\mu,\nu} - A^{\nu,\mu}) \sqrt{-g}\, X_{,\nu}\, d^3x_\mu\right] \qquad dx^\lambda \wedge d^3x_\mu = \delta^\lambda_\mu d^4x$$

$$= \left[(A^{\mu,\nu} - A^{\nu,\mu})\sqrt{-g}\right]_{,\mu} X_{,\nu}\, d^4x$$

terms with $X_{,\mu\nu}$ drop out $= 0$

arbitrary

$(\sqrt{-g} = 1 \quad \text{(Lorentz)})$
fixed

$$(A^{\mu\nu} - A^{\nu\mu})_{,\nu} = 0$$

$$F^{\mu\nu}{}_{,\nu} = 0$$

Maxwell

$$\boxed{F^{\mu\nu}{}_{;\nu} = 0}$$

covariant derivative

action: vertically transitive

Topological field theory

$$\mathcal{L} = \frac{1}{2}\, \varepsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho\, d^3x \qquad = F \wedge A$$

Diff(X) - invariant ~~~ intrinsic expression

notice that: $\int \gamma^* \mathcal{L} = \int \mathcal{L}$

$y = y(x) \equiv \gamma(x)$ $\qquad$ ($\gamma^* \mathcal{L} = \mathcal{L} + d\varphi$) in principle $\int d\varphi = 0$

$$\int (\gamma^* \mathcal{L}) = \int L(y(x))\, \underset{\sim}{J\, d^3x}$$

$$= \int L(y)\, d^3y = \int \mathcal{L}$$

$$\boxed{\delta_{(\xi, \chi)} L = \frac{1}{2}\, \varepsilon^{\mu\nu\sigma} F_{\mu\nu}\, \chi_{,\sigma}}$$

Noether:

$$d\,(j^1 A)\, J^{\mathcal{L}}(\xi, \chi) = \underset{\substack{\| \\ 0}}{\varepsilon^{\mu\nu\sigma} F_{\mu\nu}} \left(A_{\sigma,\tau}\, \xi^{\tau} + A_{\tau}\, \xi^{\tau}_{,\sigma} + \frac{1}{2}\chi_{,\sigma} \right)$$

CS: $F_{\mu\nu} = 0$

also:

$$d\left((j^1\phi)^* J^{\mathcal{L}}(\xi) \right) = \left\{ \underset{\substack{\| \\ 0}}{\frac{\delta L}{\delta \phi^A}} (\mathcal{L}_\xi \phi)^A + \underset{\substack{\| \\ 0}}{\delta_\xi L} \right\} (j^1\phi)\, d^{n+1}x$$

CS $\qquad$ on shell

M. Spera UCSC. Brescia

- 2nd order jet bundle
- Lepage & De Donder-Weyl operator
- Hamilton multisympl. geometry

★ On the second order jet bundle

(accommodating second derivatives)

(†) keep track of $\partial_i \partial_j \varphi = \partial_j \partial_i \varphi$

$$J^1(J^1E) \sim (x^\mu, q^i, q^i_\mu, r^i_\mu, q^i_{\mu\rho})$$

jetify jetify

restrict :

$$\boxed{\begin{array}{ll} (\blacklozenge) & q^i_\mu = r^i_\mu \quad (\text{"semiholonomicity"}) \\ (\blacklozenge\blacklozenge) & q^i_{\mu\rho} = q^i_{\rho\mu} \end{array}}$$

(cf. (†))

coordinate transformations are worked out, and they preserve $(\blacklozenge)$ and $(\blacklozenge\blacklozenge)$ ⇝ get J^2E (in the notation of Forger–Romero)

one has a symmetric/antisymmetric decomposition

$q^j_{(\nu\sigma)}$ clumsy $\qquad q^j_{[\nu\sigma]}$ simple

One mainly needs

$$j^2\varphi(x) = (x^\mu, \varphi^i(x), \partial_\mu \varphi^i(x), \partial_\mu \partial_\nu \varphi^i(x))$$

In [Forger-Romero] : ⊛ linear dual ⊛ affine dual

$J^1E \to E$ affine $\qquad \vec{J}^1E$: difference vector bundle

multiphase space : $\quad J^1\circledast E$ twisted linear dual of $\vec{J}^1E$

extended m. space $\quad J^1\circledast E$ twisted affine dual of JE

$\mathcal{L} : J^1E \to \Lambda^n M \quad$ ⟵ ⚠ different from GIMMSY ($\begin{array}{ccc} X^{m+1} & \to & M^n \\ Y & \to & E \end{array}$)

$\mathcal{H}$: a section of $J^1\circledast E \to \vec{J}^1\circledast E$

De Donder – Weyl Hamiltonian $\qquad \mathcal{H} = -H\, d^n x$

$$\mathcal{H} = \mathbb{F}\mathcal{L} \circ (\vec{\mathbb{F}}\mathcal{L})^{-1}$$

affine Legendre $\qquad$ linear Legendre $\qquad$ hyperregularity condition

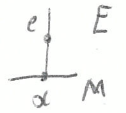
$\mathcal{L} = L\, d^n x \qquad \mathcal{H} = -H\, d^n x$

$H = p^\mu_i q^i_\mu - L$

★ Details

Known ⟶ extended Legendre

$$\mathbb{F}\mathcal{L}(\gamma)\cdot \kappa = \mathcal{L}(\gamma) + \frac{d}{d\lambda}\mathcal{L}(\gamma+\lambda(\kappa-\gamma))\Big|_{\lambda=0} \qquad \kappa \in J^1_e E$$



ordinary Legendre

$$\vec{\mathbb{F}}\mathcal{L}(\gamma)\cdot\vec{\kappa} = \frac{d}{d\lambda}\mathcal{L}(\gamma+\lambda\vec{\kappa})\Big|_{\lambda=0} \qquad \vec{\kappa} \in \vec{J}^1_e E$$

One has an obvious map η from extended to ordinary Legendre; symbolically

$$\vec{\mathbb{F}}\mathcal{L} = \eta\circ\mathbb{F}\mathcal{L}$$

$$\boxed{\mathbb{F}\mathcal{L}: \quad \underbrace{p^\mu_i = \frac{\partial L}{\partial q^i_\mu}}_{\vec{\mathbb{F}}\mathcal{L}}, \quad p = L - \frac{\partial L}{\partial q^i_\mu}\, q^i_\mu \qquad \mathcal{L} = L\, d^n x}$$

Let $\vec{\mathbb{F}}\mathcal{L}$ be a diffeomorphism (hyperregularity) (⚠ not valid in presence of symmetries)

$$\boxed{\mathcal{H} := \mathbb{F}\mathcal{L}\circ\left(\vec{\mathbb{F}}\mathcal{L}\right)^{-1}} \qquad \boxed{\mathcal{H}\circ\vec{\mathbb{F}}\mathcal{L} = \mathbb{F}\mathcal{L}}$$

☆ De Donder Weyl Hamiltonian

$$\mathcal{L} = L\, d^n x \qquad \mathcal{H} = -H\, d^n x \qquad H = p^\mu_i q^i_\mu - L$$

★ Hamiltonian formulation

Start from $\mathcal{H}$ (section of $J^{1\circledast}E \longrightarrow \vec{J}^{1\circledast}E$); define then

$$\mathbb{F}\mathcal{H}: \quad \vec{J}^{1\circledast}E \longrightarrow J^1 E$$

$$q^i_\mu := \frac{\partial H}{\partial p^\mu_i} \qquad (\rightsquigarrow \quad L = p^\mu_i q^i_\mu - H)$$

One may choose to work directly within Hamilton's framework

$\star$ Canonical forms

$$\theta = p^\mu_i\, dq^i \wedge d^n x_\mu + p\, d^n x$$

$$\omega = -d\theta = dq^i \wedge dp^\mu_i \wedge d^n x_\mu - dp \wedge d^n x$$

⚠ we are now adhering to Forger-Romero on

$J^1 \circledast E$

$\downarrow \quad \uparrow \mathcal{H}$

$\vec{J}^1 \circledast E$

Poincaré-Cartan : $\Theta_\mathcal{L}$, $\omega_\mathcal{L}$ on $J^1 E$

$$\Theta_\mathcal{L} = (\mathbb{F}\mathcal{L})^* \theta \quad , \quad \omega_\mathcal{L} = (\mathbb{F}\mathcal{L})^* \omega$$

De Donder-Weyl : $\Theta_\mathcal{H}$, $\omega_\mathcal{H}$ on $\vec{J}^1 \circledast E$

$$\Theta_\mathcal{H} = \mathcal{H}^* \theta \quad , \quad \omega_\mathcal{H} = \mathcal{H}^* \omega$$

Assuming $\mathcal{H} \circ \vec{\mathbb{F}}\mathcal{L} = \mathbb{F}\mathcal{L}$ we get the old expression for $\Theta_\mathcal{L}$, $\omega_\mathcal{L}$

$$\Theta_\mathcal{L} = \frac{\partial L}{\partial q^i_\mu} dq^i \wedge d^n x_\mu + \left(L - \frac{\partial L}{\partial q^i_\mu} q^i_\mu\right) d^n x \qquad \omega_\mathcal{L} = -d\Theta_\mathcal{L}$$

$$\Theta_\mathcal{H} = p^\mu_i\, dq^i \wedge d^n x_\mu - H\, d^n x$$

$$\omega_\mathcal{H} = -d\Theta_\mathcal{H} = dq^i \wedge dp^\mu_i \wedge d^n x_\mu - dH \wedge d^n x$$

Actions:

$$\underset{\Gamma(E)}{S[\varphi]} = \int_{j^1\varphi} (\varphi, \partial\varphi)^* \Theta_\mathcal{L}$$

$$\underset{\Gamma(\vec{J}^1 \circledast E)}{S[\varphi, \pi]} = \int [(\varphi, \pi), (\partial\varphi, \partial\pi)]^* \Theta_\mathcal{H}$$

$$\begin{cases} q^i = \varphi^i(x) \\ p^\mu_i = \pi^\mu_i(x) \end{cases}$$

★ The Euler Lagrange map & the $\mathcal{E}$-$\mathcal{L}$ equations

(revisited à la Forger-Romero)

$$D_{\mathcal{L}} : \quad J^2 E \longrightarrow V^{\circledast} E$$

affine vertical bundle (dual)

Symmetric 2nd order jet space

$$\underbrace{(\varphi, \partial\varphi, \partial^2\varphi)}_{j^2\varphi} \longmapsto \underbrace{(\varphi, \partial\varphi)^*}_{(j^1\phi)^*} \left(i_{\tilde{V}} \omega_{\mathcal{L}} \right)$$

projecting to V

Actually $\tilde{V} = j^1 V$

Cf. GIMSY

$$\parallel\parallel$$

$$D_{\mathcal{L}}(\varphi, \partial\varphi, \partial^2\varphi) \cdot V$$

Then $$\boxed{D_{\mathcal{L}} = 0 \iff \varphi \text{ satisfies } \mathcal{E}\text{-}\mathcal{L}}$$

This is exactly the calculation (even simpler...) we performed it in a previous lecture.

✶ The De Donder-Weyl map & the (H-V-DD-W) equations

Hamilton, Volterra, De Donder, Weyl

vertical part (see below for details)

$$\mathcal{D}_{\mathcal{H}} : J^1(\overset{\circ}{J}{}^{1\circledast}E) \longrightarrow V^{\circledast}(\overset{\circ}{J}{}^{1\circledast}E)$$

$\overset{\circ}{J}{}^{1\circledast}E \to M$

$$\boxed{\mathcal{D}_{\mathcal{H}}(\varphi, \pi, \partial\varphi, \partial\pi)\cdot \mathbf{V} := (\varphi,\pi)^*(i_{\mathbf{V}}\omega_{\mathcal{H}})}$$

vertical vector

Then: $\mathcal{D}_{\mathcal{H}} = 0 \iff (\varphi,\pi)$ satisfies the H-V-DD-W equations

Let us prove this.

$$\omega_{\mathcal{H}} = dq^i \wedge dp_i^\mu \wedge d^n x_\mu + dH \wedge d^n x$$

$$V = v^i \frac{\partial}{\partial q^i} + V_i^\mu \frac{\partial}{\partial p_i^\mu} \quad \text{(vertical)}$$

Compute $i_V \omega_{\mathcal{H}}$:

2nd place

$$i_V\omega_{\mathcal{H}} = v^i \, dp_i^\mu \wedge d^n x_\mu + \frac{\partial H}{\partial q^i} v^i d^n x - \underbrace{V_i^\mu \partial_\nu \varphi^i \, dx^\nu \wedge d^n x_\mu}_{= -V_i^\mu \partial_\mu \varphi^i d^n x} + V_i^\mu \frac{\partial H}{\partial p_i^\mu} d^n x$$

as before

$dp_i^\mu = \partial_\nu \pi_i^\mu dx^\nu$, $\; v^i \, dp_i^\mu \wedge d^n x_\mu = v^i \partial_\mu \pi_i^\mu d^n x$

assembling pieces :

$\boxed{\mathcal{D}_{\mathcal{H}}}\,(\varphi, \pi, \partial\varphi, \partial\pi)$ (section) as an operator

"divergence"

$$\boxed{= \left(\frac{\partial H}{\partial q^i}(\varphi,\pi) + \partial_\mu \pi^\mu_i\right) dq^i \otimes d^n x + \left(\frac{\partial H}{\partial p^\mu_i}(\varphi,\pi) - \partial_\mu \varphi^i\right) dp^\mu_i \otimes d^n x}$$

$\leadsto$ H-V-DD-W

$$\boxed{\begin{cases} \dfrac{\partial H}{\partial q^i} + \partial_\mu \pi^\mu_i = 0 \\[2ex] \dfrac{\partial H}{\partial p^\mu_i} - \partial_\mu \varphi^i = 0 \end{cases}}$$

* Remark

GIMSY	FR	Hélein et al, Krijowski Szczyrba..	Rovelli
they privilege a strictly lagrangian view point	both Lagrangian & Hamiltonian viewpoints	insist on a Hamiltonian framework	Palatini formalism (i.e. tetradic) for General Relativity)

The Hamilton - Volterra - De Donder - Weyl equations (HVDDW) (simplified) (see Hélein 2011)

ASIDE

Lecture XIII - bis

$$H = H(x, \underbrace{u(x), p(x)}_{\varphi(x)\ \text{or}\ p^*(x)})$$

$x \in X$, $u(x) \in Y$, $p(x) \in \mathrm{End}(X,Y)^* = \mathrm{End}(Y^*, X^*)$, $p = p_i^\mu$

X, Y f.d. vector spaces

$$\boxed{\begin{cases} \dfrac{\partial H}{\partial y^i} = -\sum_\mu \dfrac{\partial p_i^\mu}{\partial x^\mu} \\[2ex] \dfrac{\partial u}{\partial x^\mu} = \dfrac{\partial H}{\partial p^\mu} \end{cases}}$$

$u : X \to Y$

β = vol form on X

$\beta = dx^1 \wedge \dots dx^n$

☆ Equivalent formulation

Einstein convention

$$(\overset{1}{dp_i^\mu} \wedge \overset{1}{dy^i} \wedge \overset{n-1}{\beta_\mu})(\xi, \overset{n+1}{X_1 \dots X_n})$$

local basis

$$\beta_\mu \equiv \partial_\mu \lrcorner \beta$$

$$= (\overset{1}{dH} \wedge \overset{n}{\beta})(\xi, X_1 \dots X_n)$$

sum over ν & j

$$X_\mu = \frac{\partial}{\partial x^\mu} + \frac{\partial u^i}{\partial x^\mu}\frac{\partial}{\partial y^i} + \frac{\partial p_j^\nu}{\partial x^\mu}\frac{\partial}{\partial p_j^\nu}$$

basis for $T_\cdot(\Gamma = \mathrm{Im}\,\varphi)$ $\quad \varphi : x \mapsto (x, u(x), p(x))$

get a submanifold $\Gamma = \mathrm{Im}\,\varphi$

$$\underbrace{[dp_i^\mu \wedge dy^i \wedge \beta_\mu - dH \wedge d\beta]}_{\Omega}(\xi, \dots) = 0$$

$(n+1)$-form

$$\Omega\big|_\Gamma (\xi, X_1 \dots X_n) = 0 \quad \forall \xi$$

Special cases

$\boxed{n = 1}$

$x = t$

$y = x = q$ $\qquad \beta = dt$

$$X_1 = \frac{\partial}{\partial t} + \dot q \frac{\partial}{\partial q} + \dot p \frac{\partial}{\partial p}$$

$$\partial_\mu = \partial_t \equiv \frac{\partial}{\partial t}$$

$$\left(\beta, \frac{\partial}{\partial t}\right) = 1$$

$$\Big[\, dp \wedge dq\, (\xi, X_1)$$

$$\|$$

$$dp(\xi)\, dq(X_1) - dp(X_1)\, dq(\xi)$$

$$= dp(\xi) \cdot \dot q - \dot p\, dq(\xi)\,\Big]$$

$$\Big[\, (dH \wedge \beta)(\xi, X_1) = dH(\xi)\,\beta(X_1) - dH(X_1)\,\beta(\xi)$$

$$= dH(\xi) - \left(\frac{\partial H}{\partial t} + \dot q \frac{\partial H}{\partial q} + \dot p \frac{\partial H}{\partial p}\right)\beta(\xi)\,\Big]$$

$$=$$

$$\Big[\, \dot q\, dp - \dot p\, dq \overset{?}{=} \underbrace{dH}_{\substack{\| \\ \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp}} - \left(\frac{\partial H}{\partial t} + \dot q \frac{\partial H}{\partial q} + \dot p \frac{\partial H}{\partial p}\right) dt \,\Big]$$

$$\overset{?}{=} \underbrace{-\left(\dot q \frac{\partial H}{\partial q} + \dot p \frac{\partial H}{\partial p}\right)}_{(*)} dt + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp$$

iff: $\Big[\, \dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q} \,\Big]$ $\quad (\Rightarrow (*) = 0)$

★ Hamilton

Another special case $\qquad n = 2 \qquad \dim Y = 1$

$$X_\mu = \frac{\partial}{\partial x^\mu} + \frac{\partial u}{\partial x^\mu}\frac{\partial}{\partial y} + \frac{\partial p^1}{\partial x^\mu}\frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^\mu}\frac{\partial}{\partial p^2} \qquad \mu = 1, 2$$

$$\beta = dx^1 \wedge dx^2 \qquad \beta_1 = dx^2 \qquad \beta_2 = -dx^1$$

$$(\overbrace{dp^1 \wedge dy \wedge dx^2}^{1'} - \overbrace{dp^2 \wedge dy \wedge dx^1}^{1''})(\xi, X_1, X_2)$$

$$\stackrel{?}{=} (dH \wedge \underbrace{\beta}_{dx_1 \wedge dx_2})(\xi, X_1, X_2)$$

Compute the l. h. s.

1' $\quad (dp^1 \wedge dy \wedge dx^2)(\xi, X_1, X_2) =$

$$= \begin{vmatrix} dp^1(\xi) & dy(\xi) & dx^2(\xi) \\ dp^1(X_1) & dy(X_1) & dx^2(X_1) \\ dp^1(X_2) & dy(X_2) & dx^2(X_2) \end{vmatrix} = \begin{vmatrix} dp^1(\xi) & dy(\xi) & dx^2(\xi) \\ \frac{\partial p^1}{\partial x^1} & \frac{\partial u}{\partial x^1} & 0 \\ \frac{\partial p^1}{\partial x^2} & \frac{\partial u}{\partial x^2} & 1 \end{vmatrix} =$$

$$= \frac{\partial u}{\partial x^1} dp^1(\xi) - \frac{\partial p^1}{\partial x^1} dy(\xi) + \underbrace{\left(\frac{\partial p^1}{\partial x^1}\frac{\partial u}{\partial x^2} - \frac{\partial u}{\partial x^1}\frac{\partial p^1}{\partial x^2}\right)}_{\{p^1, u\}} dx^2(\xi)$$

1''

$$\begin{vmatrix} dp^2(\xi) & dy(\xi) & dx^1(\xi) \\ dp^2(x_1) & dy(x_1) & dx^1(x_1) \\ dp^2(x_2) & dy(x_2) & dx^1(x_2) \end{vmatrix} =$$

$$\begin{vmatrix} dp^2(\xi) & dy(\xi) & dx^1(\xi) \\ \frac{\partial p^2}{\partial x^1} & \frac{\partial u}{\partial x^1} & 1 \\ \frac{\partial p^2}{\partial x^2} & \frac{\partial u}{\partial x^2} & 0 \end{vmatrix} = -\frac{\partial u}{\partial x^2}\, dp^2(\xi) + \frac{\partial p^2}{\partial x^2}\, dy(\xi) + \underbrace{\left(\frac{\partial p^2}{\partial x^1}\frac{\partial u}{\partial x^2} - \frac{\partial p^2}{\partial x^2}\frac{\partial u}{\partial x^1}\right)}_{\{p^2, u\}} dx^1(\xi)$$

l.h.s.

1' $-$ 1'' (ξ omitted)

$$= -\left(\frac{\partial p^1}{\partial x^1} + \frac{\partial p^2}{\partial x^2}\right) dy + \frac{\partial u}{\partial x^1}\, dp^1 + \frac{\partial u}{\partial x^2}\, dp^2 + \{p^1, u\}\, dx^2 - \{p^2, u\}\, dx^1$$

l.h.s.

r. h. s.

$$2 = (dH \wedge dx^1 \wedge dx^2)(\xi, X_1, X_2) =$$

$$dH = \frac{\partial H}{\partial x^1} dx^1 + \frac{\partial H}{\partial x^2} dx^2 + \frac{\partial H}{\partial y} dy + \frac{\partial H}{\partial p^1} dp^1 + \frac{\partial H}{\partial p^2} dp^2$$

$$\underbrace{\frac{\partial H}{\partial y} dy \wedge dx^1 \wedge dx^2}_{2'} + \underbrace{\frac{\partial H}{\partial p^1} dp^1 \wedge dx^1 \wedge dx^2}_{2''} + \overbrace{\frac{\partial H}{\partial p^2} (dp^2 \wedge dx^1 \wedge dx^2)}^{2'''} (\xi, X_1, X_2)$$

$$2': \quad \frac{\partial H}{\partial y} \begin{vmatrix} dy(\xi) & dx^1(\xi) & dx^2(\xi) \\ dy(X_1) & dx^1(X_1) & dx^2(X_1) \\ dy(X_2) & dx^1(X_2) & dx^2(X_2) \end{vmatrix} = \frac{\partial H}{\partial y} \begin{vmatrix} dy(\xi) & dx^1(\xi) & dx^2(\xi) \\ \frac{\partial u}{\partial x^1} & 1 & 0 \\ \frac{\partial u}{\partial x^2} & 0 & 1 \end{vmatrix}$$

$$= \frac{\partial H}{\partial y} \left(dy(\xi) - \frac{\partial u}{\partial x^1} dx^1(\xi) - \frac{\partial u}{\partial x^2} dx^2(\xi) \right)$$

$$\Rightarrow \text{gilt} \quad \frac{\partial H}{\partial y} = -\left(\frac{\partial p^1}{\partial x^1} + \frac{\partial p^2}{\partial x^2} \right) \quad \checkmark$$

$$2'' \quad \frac{\partial H}{\partial p^1} \begin{vmatrix} dp^1(\xi) & dx^1(\xi) & dx^2(\xi) \\ dp^1(X_1) & dx^1(X_1) & dx^2(X_1) \\ dp^1(X_2) & dx^1(X_2) & dx^2(X_2) \end{vmatrix} = \frac{\partial H}{\partial p^1} \begin{vmatrix} dp^1(\xi) & dx^1(\xi) & dx^2(\xi) \\ \frac{\partial p^1}{\partial x^1} & 1 & 0 \\ \frac{\partial p^1}{\partial x^2} & 0 & 1 \end{vmatrix} =$$

$$= \frac{\partial H}{\partial p^1} \left(dp^1(\xi) - \frac{\partial p^1}{\partial x^1} dx^1(\xi) - \frac{\partial p^1}{\partial x^2} dx^2(\xi) \right)$$

$\textcircled{2'''}$:

$$\frac{\partial H}{\partial p^2}\begin{vmatrix} dp^2(\xi) & dx^1(\xi) & dx^2(\xi) \\ dp^2(x_1) & dx^1(x_1) & dx^2(x_1) \\ dp^2(x_2) & dx^1(x_2) & dx^2(x_2) \end{vmatrix} = \frac{\partial H}{\partial p^2}\begin{vmatrix} dp^2(\xi) & dx^1(\xi) & dx^2(\xi) \\ \frac{\partial p^2}{\partial x^1} & 1 & 0 \\ \frac{\partial p^2}{\partial x^2} & 0 & 1 \end{vmatrix}$$

$$= \frac{\partial H}{\partial p^2}\left[dp^2(\xi) - \frac{\partial p^2}{\partial x^1} dx^1(\xi) - \frac{\partial p^2}{\partial x^2} dx^2(\xi) \right]$$

$$\textcircled{2} = \textcircled{2'} + \textcircled{2''} + \textcircled{2'''} \qquad \left(\stackrel{?}{=} \textcircled{1'} - \textcircled{1''} = \textcircled{1} \right)$$

$$\frac{\partial u}{\partial x^1} = \frac{\partial H}{\partial p^1}, \quad \frac{\partial u}{\partial x^2} = \frac{\partial H}{\partial p^2} \qquad \checkmark$$

$\Rightarrow$ check terms in dx^1, dx^2

$\boxed{dx^1}$ in $\textcircled{1}$: $\quad -\{p^2, u\} = +\{u, p^2\}$

in $\textcircled{2}$: $\quad -\frac{\partial H}{\partial u}\frac{\partial u}{\partial x^1} - \frac{\partial H}{\partial p^1}\frac{\partial p^1}{\partial x^1} - \frac{\partial H}{\partial p^2}\frac{\partial p^2}{\partial x^1}$

$$= \left(\frac{\partial p_1}{\partial x^1} + \frac{\partial p^2}{\partial x^2}\right)\frac{\partial u}{\partial x^1} - \frac{\partial u}{\partial x^1}\frac{\partial p^1}{\partial x^1} - \frac{\partial u}{\partial x^2}\frac{\partial p^2}{\partial x^1}$$

cancel out

$$= \frac{\partial p^2}{\partial x^2}\frac{\partial u}{\partial x^1} - \frac{\partial u}{\partial x^2}\frac{\partial p^2}{\partial x^1} = \{u, p^2\} \qquad \checkmark$$

$\boxed{dx^2}$ in ① $\{p^1, u\}$

in ②

$$-\frac{\partial H}{\partial y}\frac{\partial u}{\partial x^2} - \frac{\partial H}{\partial p^1}\frac{\partial p^1}{\partial x^2} - \frac{\partial H}{\partial p^2}\frac{\partial p^2}{\partial x^2} =$$

$$= \left(\frac{\partial p_1}{\partial x^1} + \frac{\partial p_2}{\partial x^2}\right)\frac{\partial u}{\partial x^2} - \frac{\partial u}{\partial x^1}\frac{\partial p^1}{\partial x^2} - \frac{\partial u}{\partial x^2}\frac{\partial p^2}{\partial x^2}$$

cancel out

$$= \frac{\partial p_1}{\partial x^1}\frac{\partial u}{\partial x^2} - \frac{\partial u}{\partial x^1}\frac{\partial p^1}{\partial x^2} = \{p^1, u\}$$

✓

+ Variant

extra variable "dual" to β

$$\omega = de \wedge \beta + dp_i^\mu \wedge dy \wedge \beta_\mu$$

$$\Gamma = \{x, u, e, p^*\}$$

$$\Gamma^* = \{x, u, p^*\} \qquad \beta|_{\Gamma^*} \neq 0$$

$$X = X_{old} + \frac{\partial e}{\partial x}\frac{\partial}{\partial e}$$ ($x = t$ single variable)

$$\boxed{\omega(\xi, \dots)|_\Gamma = (d\mathcal{H} \wedge \beta)(\xi, \dots)|_\Gamma \quad \forall \xi}$$

$$\Omega = \omega - d\mathcal{H} \wedge \beta$$

$$\Omega(\xi, \dots)|_\Gamma = 0 \quad \forall \xi$$

$\Leftrightarrow$ HV-DD-W

"solution of the H-V-DD-W system"

$$\mathcal{H} = e + H(x, y, p^*) = h \quad \text{on } \Gamma$$

$$(X \in T_\cdot\Gamma \text{ iff } d\mathcal{H}(X) = 0)$$

$$(*) \quad (d\mathcal{H} \wedge \beta)(\xi, X_1 \dots X_n) = d\mathcal{H}(\xi)\,\beta(X_1 \dots X_n)$$

$\cap T\Gamma$ (+ terms = 0)

Check: (n=1)

$$[(de \wedge \overset{dt}{\beta} + dp \wedge dy)(\xi, X) = \qquad X = X_{old} + \dot{e}\frac{\partial}{\partial e}$$

$$= de(\xi)\,dt(X) - de(X)\,dt(\xi) + dp(\xi)\,dy(X) - dp(X)\,dy(\xi)$$

$$= de(\xi) - \dot{e}\,dt(\xi) + dp(\xi)\,\dot{q} - \dot{p}\,dq(\xi)$$

$$\equiv de - \dot{e}\,dt + \dot{q}\,dp - \dot{p}\,dq]$$

on Γ $\quad de = \dot{e}\,dt$

$$[(d\mathcal{H} \wedge dt)(\xi, X) \underset{(*)}{=} d\mathcal{H}(\xi)\underbrace{dt(X)}_{1} = d\mathcal{H}(\xi)$$

$$d\mathcal{H} = de + dH = \dot{e}\,dt + \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp$$

$$= \underbrace{\left(\dot{e} + \frac{\partial H}{\partial t}\right)}_{0}dt + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp]$$

on Γ: $\mathcal{H} = h \Rightarrow \dot{e} + \frac{\partial H}{\partial t} = 0$

$$\Rightarrow \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Back to the case $n=2$ (different notation)

Krijovski 73

$$W = M \times \mathbb{R}$$

KLEIN-GORDON

$$(x^0, x^1, \varphi) \qquad p \mapsto \eta$$

Let $H = H(x^\mu, \varphi, \eta^\mu) = -\frac{1}{2}\left(\underbrace{\eta_\mu \eta^\mu}_{g_{\mu\nu}\eta^\mu\eta^\nu} + m^2\varphi^2\right)$

$$\eta_\mu = g_{\mu\nu}\eta^\nu \qquad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{minkowski}$$

Resume H-V-DD-W [sign conventions Klein ≠ Krijovski] ‼ (*)

$$\frac{\partial H}{\partial \eta^{\mu_0}} = -\frac{1}{2} \cdot g_{\mu\nu}\left[\eta^\nu \delta^{\mu\mu_0} + \eta^\mu \delta^{\nu\mu_0}\right]$$

$$= -\frac{1}{2} \cdot \underbrace{g_{\mu\nu}\eta^\nu}_{\eta_\mu} \cdot \delta^{\mu\mu_0} - \frac{1}{2}\underbrace{g_{\mu\nu}\eta^\mu}_{\eta_\nu} \cdot \delta^{\nu\mu_0}$$

$$\eta_\nu \cdot \delta^{\nu\mu_0}$$

$$= -\frac{1}{2}\left(\eta_{\mu_0} + \eta_{\mu_0}\right) = -\eta_{\mu_0}$$

$$\Rightarrow \frac{\partial \varphi}{\partial x^\mu} = \eta_\mu \qquad \text{(valid in general)}$$

together with (*) $\sum_\mu \frac{\partial \eta^\mu}{\partial x^\mu} = \frac{\partial H}{\partial \varphi} = -\frac{1}{2} m^2 \cdot 2\varphi = -m^2\varphi$

$$\frac{\partial \eta^\mu}{\partial x^\mu} = \partial_\mu \eta^\mu = \partial^\mu \eta_\mu = \partial^\mu \partial_\mu \varphi$$

$$\Rightarrow \boxed{(\partial^\mu \partial_\mu + m^2)\varphi = 0} \qquad \text{Klein-Gordon}$$

Comparison of notation $\qquad \eta^i \equiv p^i$

old : (rewritten) Śclein

$$\Omega = d\eta^0 \wedge d\varphi \wedge dx^1 - d\eta^1 \wedge d\varphi \wedge dx_0 - dH \wedge dx^0 \wedge dx^1$$

new : Kijowski

$$d\omega = dH \wedge dx^0 \wedge dx^1 + d\eta^0 \wedge d\varphi \wedge dx^1 + d\eta^1 \wedge d\varphi \wedge dx^0$$

difference : overall sign + relative sign

★ Electrodynamics (Kijowski)

$W = T^*M \quad M$ space-time

$A = A_\mu dx^\mu$ (el. potential)

$\wp$: constraints $h = H(x^\mu, A_\mu, h^{\mu\nu}) \quad h^{\mu\nu} = -h^{\nu\mu}$

$$\omega = h\, dx^0 \wedge \dots dx^3 + h^{\mu\nu} dx^0 \wedge \dots \underbrace{dA_\mu}_{\nu} \wedge \dots dx^3$$

$$\gamma = d\omega|_\wp = dH \wedge dx^0 \dots dx^3 + dh^{\mu\nu} dx^0 \wedge \underbrace{dA_\mu}_{\nu} \wedge dx^3$$

DD-W-Equations

$$\boxed{\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &= \frac{\partial H}{\partial h^{\mu\nu}} \\ \partial_\nu h^{\mu\nu} &= \frac{\partial H}{\partial A_\mu} \end{aligned}}$$

Set $\dfrac{\partial H}{\partial h^{\mu\nu}} = f_{\mu\nu}$; $\dfrac{\partial H}{\partial A_\mu} = j^\mu$

strength

get n.l. electrodynamics:
$$\begin{cases} \partial_\mu A_\nu - \partial_\nu A_\mu = f_{\mu\nu} \\ \partial_\nu h^{\mu\nu} = j^\mu \end{cases}$$

★ Maxwell without currents:

$$\boxed{H = \frac{1}{4} h^{\mu\nu} h_{\mu\nu}} \quad , \; f_{\mu\nu} = \frac{\partial H}{\partial h^{\mu\nu}} = h_{\mu\nu} \quad \boxed{, \; j^\mu = 0}$$

$$\boxed{\begin{cases} \partial_\mu A_\nu - \partial_\nu A_\mu = f_{\mu\nu} \\ \partial_\nu f^{\mu\nu} = 0 \end{cases}}$$

★★★ The Euler equation

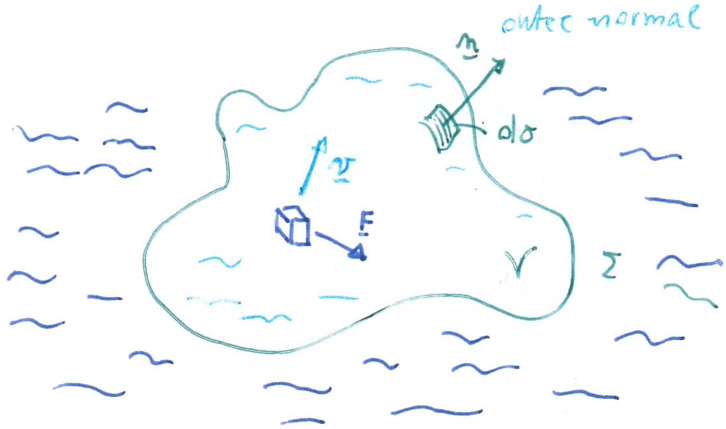
TOPICS IN SYMPLECTIC AND MULTISYMPLECTIC GEOMETRY
Ph.D. Course

M. Spera UCSC - Brescia

Lecture XIV

- Euler equation (Riemannian geom. interpretation)

Derivation from dynamical principles



$\underline{F}$: force acting on the fluid (per unit mass)

Σ bounding a domain $\mathcal{V}$

force exerted by surrounding fluid on surface element $d\sigma$: $-p\,\underline{n}\cdot d\sigma$ (p : pressure (Pascal's principle))

Total force exerted on Σ : $-\int_\Sigma p\cdot\underline{n}\,d\sigma = -\int_{\mathcal{V}} \nabla p\, dv$ (vector equation) (dv: volume element)

Apply Newton's law "$\underline{F} = m\,\underline{a}$" ($\rho$: density)

($d(p\,\omega) = dp\wedge\omega + p\,d\omega$, ω: area 2-form, $d\omega = 0$, $= dp\wedge\omega$)

$$\boxed{\int_{\mathcal{V}} (\rho\underline{F} - \nabla p)\,dv = \int_{\mathcal{V}} \rho \frac{d\underline{v}}{dt}\,dv}$$

$\Rightarrow$ ($\mathcal{V}$ is arbitrary)

$$\frac{d\underline{v}}{dt} = \underline{F} - \frac{1}{\rho}\nabla p$$

Let us set $\rho = 1$ (homogeneity), $\underline{F} \equiv \underline{0}$; we get $\frac{d\underline{v}}{dt} = -\nabla p$, i.e.

$$\boxed{\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\nabla\,\underline{v} = -\nabla p}$$

Euler equation (homogeneous, inviscid fluid)

intrinsically:

$$\boxed{\frac{\partial v}{\partial t} + \nabla_v v = -\nabla p}$$

Levi-Civita connection

∇p: Riemannian gradient

✶ Balance equation

$\mathcal{G}$: density of a scalar quantity transported by $\underline{j}$

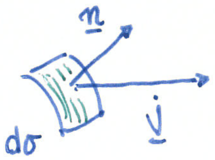
γ : production rate of $\mathcal{G}$ per unit volume & time

One gets

$$(\star) \quad \boxed{\frac{\partial \mathcal{G}}{\partial t} + \operatorname{div} \underline{j} = \gamma}$$

$\gamma > 0$ source

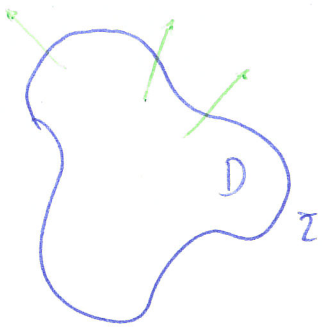
$\gamma < 0$ sink



$\underline{j} \cdot \underline{n}$: amount transported per unit time & area (flux density)

Proof :

$$\boxed{\frac{\partial}{\partial t} \int_D \mathcal{G} \, d^3x} = \int_D \frac{\partial \mathcal{G}}{\partial t} \, d^3x$$



$$= \int_D \gamma \, d^3x - \underbrace{\int_\Sigma \underline{j} \cdot \underline{n} \, d\sigma}_{\text{ingoing flux}}$$

production rate of $\mathcal{G}$ in D

$$\underset{\text{(divergence theorem)}}{=} \boxed{\int_D (\gamma - \operatorname{div} \underline{j}) \, d^3x} \Rightarrow (\star)$$

(continuity + arbitrariness of D)

In particular, take $\mathcal{G} = \rho$ (mass density)

$\underline{j} = \rho \underline{v}$ (mass current) and get

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0}$$

✶ continuity equation (mass conservation)

If $\underline{j} = \underline{v}$ $\qquad$ $\operatorname{div} \underline{v}$ represents the rate of volume variation $\left(\mathcal{L}_v d^3x = (\operatorname{div} \underline{v})\, d^3x \right)$

(Euclidean metric)

$$\boxed{\operatorname{div} \underline{v} = 0}$$

incompressible fluid

(solenoidal motion)

Recall:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0$$

Also notice

$$\operatorname{div}(\rho \underline{v}) = \rho \operatorname{div} \underline{v} + \nabla \rho \cdot \underline{v}$$

Now consider

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + v \nabla \rho$$

"material" or total derivative

$$\frac{d\rho}{dt} = -\operatorname{div}(\rho \cdot \underline{v}) + \nabla \rho \cdot \underline{v} = -\rho \operatorname{div} \underline{v} - \cancel{\nabla \rho \cdot \underline{v}} + \cancel{\nabla \rho \cdot \underline{v}}$$

$$\Rightarrow \qquad \frac{d\rho}{dt} + \rho \operatorname{div} \underline{v} = 0 \qquad \text{(Lagrange)}$$

For a solenoidal motion $\operatorname{div} \underline{v} = 0 \quad \Rightarrow \quad \dfrac{d\rho}{dt} \equiv 0$

$$\not\Downarrow$$

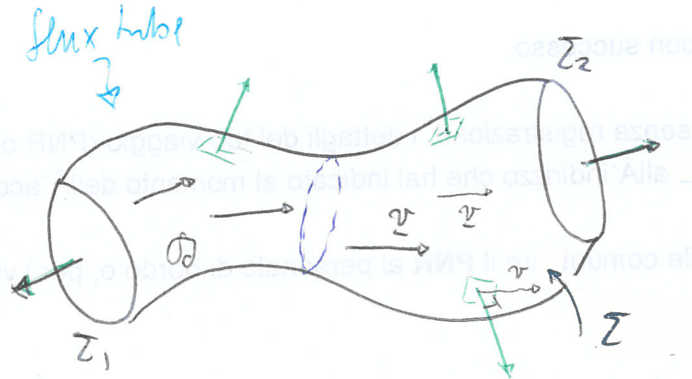
$$\rho = c$$

Conversely $\dfrac{d\rho}{dt} \equiv 0 \Rightarrow \operatorname{div} \underline{v} = 0$

* Castelli's law

In a solenoidal motion, the flux across any section of a flux tube does not depend on the section (it is called flow)

This is clear from the divergence theorem



$$
-\underbrace{\int_{\Sigma_1} \underline{v}\cdot\underline{n}\, d\sigma}_{-\Phi_1} + \underbrace{\int_{\Sigma_2} \underline{v}\cdot\underline{n}}_{\Phi_2} + \underbrace{\int_{\Sigma} \underline{v}\cdot\underline{n}\, d\sigma}_{=0}
$$

flux across Σ_1

$$
= \int_{\Omega} \operatorname{div} \underline{v}\, dv = 0 \quad \Rightarrow \quad \boxed{\Phi_1 = \Phi_2}
$$

* Riemannian geometric interpretation of the Euler equation

In general, one deals with (M,g), compact Riemannian manifold

$\Omega \subset M$ $\quad$ Ω open, with regular boundary $\partial\Omega$; the Euler equation takes the form

$$\boxed{\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p}$$

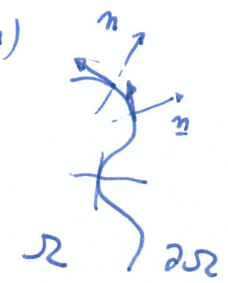
$u \in \mathfrak{X}(M)$

$p \in C^\infty(M)$

∇: Levi-Civita connection

together with conditions

$$\boxed{\begin{array}{ll} \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega \end{array}}$$



Let $F = F(t,\cdot) : \Omega \circlearrowleft$ 1-parameter family of diffeomorphisms, preserving the volume form induced by g.

volume preserving diffeomorphisms

$|||$

configuration space Q

according to the "founding fathers"

$F(0,x) = x$

spatial point

$y = F(t,x)$

material point

$y = F(x)$

$x = F^{-1}(y)$

$$u(t,y) = F_t(t,x) = \frac{\partial F}{\partial t}(t,x)$$

Eulerian velocity

$$u(t, \underbrace{F(t,x)}_{y}) = \frac{\partial F}{\partial t}(t, \underset{\substack{\| \\ F^{-1}(y)}}{x})$$

(Ebin-Marsden $\quad u = \dot\eta \circ \eta^{-1}$

$\frac{\partial F}{\partial t} \quad F^{-1}$

* Action principle

$$L(F) = \int_{t_0}^{t_1} dt \, \frac{1}{2}\int_\Omega \langle F_t, F_t \rangle \, dx$$

$F: I \times \Omega \to \Omega$

$[t_0,t_1]$

kinetic energy

metric

(+ right-invariant) $+ \; d^3\eta(x) = \underbrace{J(\eta(x))}_{1} d^3x = d^3x$

For simplicity, let us work in $\mathbb{R}^n$, with standard metric and $\bar\Omega$ compact variation: $F = F(s, t, x)$ $\quad s \in [0,1]$

variation parameter (notational abuse)

$F(0, t, x) = F(t,x)$

Set $\left.\dfrac{\partial F}{\partial s}\right|_{s=0} = v(t, F(t,x))$

$\operatorname{div} v = 0$, v tangent to $\partial\Omega$, $v = 0$ for $t = t_0, t = t_1$

Let us compute the differential of the above action:

We obtain

(★) $$\boxed{DL(F)\cdot v} = \int_t dt \int_\Omega \langle F_t(t,x), \frac{d}{dt} v(t, F(t,x)) \rangle dV$$

Heuristics: $\frac{1}{2}\langle x + \delta x, x + \delta x\rangle = \frac{1}{2}\langle x, x\rangle + \cancel{\frac{1}{2}}\cancel{2}\langle x, \delta x\rangle$
$+ \underbrace{\frac{1}{2}\langle \delta x, \delta x\rangle}_{\text{neglected}}$

More rigorously: consider the general formula

$I(x) = \int_a^b F(\underbrace{x(t)}_{x}, \underbrace{\dot{x}(t)}_{v})\, dt$, yielding, upon variation (obvious notation)

$$\left.\frac{d}{ds} I(x_s)\right|_{s=0} = \int_a^b \{F_x(x, \dot{x}) w + F_v(x, \dot{x}) \dot{w}\}\, dt$$

$w = \delta x = v$

$\dot{w} = \frac{d}{dt} v$

In our case F_x is missing (right invariance) and $F_v = \langle x, \cdot\rangle$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{F_t(t,x)}$

Resume (★): integrating by parts, and upon recalling

$v|_{t = t_0, t_1} = 0$

we get:

$$DL(F)(v) = -\int_t dt \int_\Omega \langle \frac{\partial u}{\partial t} + u \cdot \nabla_x u , v \rangle \, dv$$

gradient with respect to space variables

abstractly: $\nabla_u u$

Let us work in

closure

$$\mathbb{V} := \{ v \in C^\infty(\bar{\Omega}, T\bar{\Omega}) \,/\, \operatorname{div} v = 0, \ v \text{ tangent to } \partial\Omega \}$$

Let $P = P_{\mathbb{V}}$ actually $\bar{\mathbb{V}}^{L^2}$ (♣ Leray projector) [work in $L^2(\Omega, \mathbb{R}^n)$]

and observe that $P \frac{\partial}{\partial t} = \frac{\partial}{\partial t} P$ (♣)

Now, look for extremals

notice this

(♠) $\boxed{DL(F)(v) = 0} \quad \forall v \in \boxed{\mathbb{V}}$

(♠) becomes: $P \underbrace{\left[\frac{\partial u}{\partial t} + u \nabla_x u \right]}_{(*)} = 0$ equivalently $(1-P)(*) = (*)$

that is (remember ♣)

$$\boxed{\begin{cases} \frac{\partial u}{\partial t} + P(u \cdot \nabla u) = 0 \\ \operatorname{div} u = 0 \qquad u = Pu \\ u \,/\!/\, \partial\Omega \end{cases}}$$

Now, from compactness of $\bar{\Omega}$, and resorting to the ✱ Hodge theorem we get

$$(1-P)\mathcal{X} = \{ \nabla p \,,\; p \in H^1(\Omega) \}$$

$$\| \!\!\! | \quad L^2(\Omega, \mathbb{R}^n)$$

Sobolev space

orthogonal decomposition

$$\mathcal{V} = \mathcal{V}_0 + \nabla p$$

divergence-free ↑ ↖ gradient

$(\operatorname{div}(\nabla p) = \Delta p)$

cf Dirichlet's problem

$f \in C^2$

$$\begin{cases} \Delta f = 0 & \text{in } \Omega \\ f = f_0 & \text{on } \partial\Omega \end{cases}$$

get a unique solution

in particular, if $f_0 = 0$, we get $f = 0$

Eventually, we arrive at the

★★★ $\boxed{\text{Euler equation}}$

$$\boxed{\begin{cases} \dfrac{\partial u}{\partial t} + u \nabla u = -\nabla p \\ \operatorname{div} u = 0 \quad \text{in } \Omega \\ u \,/\!/\, \partial\Omega \end{cases}}$$

pressure

with its geometric interpretation:

$$\boxed{g = \int_M \langle \dot\eta \eta^{-1}, \dot\eta \eta^{-1} \rangle \, dv}$$

right-invariant metric on $SDiff(M)$ (volume preserving diffeomorphisms)

(change of variable formula) $d^3\eta(x) = J(\eta(x))\, d^3x = d^3x$

↪ geodesic equation for g = Euler equation

★ Remark

$$\frac{\partial u}{\partial t} + \underbrace{P(u \cdot \nabla u)}_{\text{divergence-free}} = 0$$

entails

$$0 = \operatorname{div} \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \operatorname{div} u \,,$$

therefore, if $u(0,\cdot)$ is divergence-free, so is $u(t,\cdot)$

☆☆ Additional formulations of the Euler equation

$$E: \quad \boxed{\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p}$$

$$\rightarrow E': \quad \boxed{\frac{\partial u}{\partial t} + \underbrace{\operatorname{curl} u}_{\text{vorticity}} \times u = -\nabla \Big(\underbrace{\tfrac{1}{2}|u|^2 + p}_{} \Big)}$$

★ Bernoulli function

Let us go from E' to E (work in $\mathbb{R}^3$)

Start from

$$\nabla (u \cdot v) = u \nabla v + v \nabla u + u \times \operatorname{curl} v + v \times \operatorname{curl} u$$

Set $u = v$:

$$\underset{|u|^2}{\nabla (u \cdot u)} = 2 u \nabla u + 2 u \times \operatorname{curl} u \quad \Rightarrow$$

$$\operatorname{curl} u \times u = u \nabla u - \frac{1}{2} \nabla |u|^2$$

yielding E.

If $\frac{\partial u}{\partial t} = 0$ (stationary flow), we have

$$u \times \operatorname{curl} u = \nabla \alpha \qquad \alpha = \tfrac{1}{2}|u|^2 + p$$

divergence-free   divergence-free

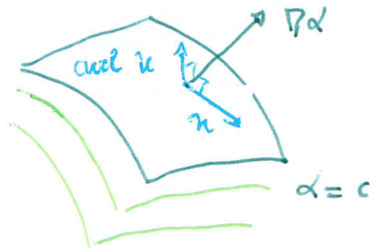
★ Bernoulli function

$$\Rightarrow \quad [\,u, \underset{\omega}{\operatorname{curl} u}\,] = 0$$

★ vorticity commutes with velocity

see also below...

structure of fluid ⤳



Lecture XV

- Euler equation (continued)

Other formulation of Euler's equation

E'' $\quad \omega = \operatorname{curl} u$

$$\boxed{\frac{\partial \omega}{\partial t} + [\omega, u] = 0}$$

!

$[\cdot,\cdot] = -$ usual bracket

E'' is obtained from the identity (for divergence-free v. fields)

$$[a,b] = \operatorname{curl}(a \times b)$$

in conjunction with E' (take its curl)

"curl" $\to$ $\displaystyle \frac{\partial u}{\partial t} + \omega \times u = -\nabla \alpha$

$$\frac{\partial \omega}{\partial t} + \operatorname{curl}(\omega \times u) = -\operatorname{curl}(\nabla \alpha) = 0$$

$$\frac{\partial \omega}{\partial t} + [\omega, u] = 0$$

Before discussing another formulation of E, in terms of differential forms, let us prove the following important identity

$$(\bigstar) \qquad \boxed{\mathcal{L}_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2}\, d\langle v, v\rangle}$$

$v \in \mathfrak{X}$ vector field

$v^\flat \in \Lambda^1$

($\sharp$, $\flat$ raising, lowering indices — musical isomorphisms)

Let $w \in \mathfrak{X}$ such that $[w, v] = 0$. Then

$$\mathcal{L}_a \langle b, c\rangle = \langle \nabla_a b, c\rangle + \langle b, \nabla_a c\rangle$$

a, b, c v. fields; $\langle\,,\rangle$ metric

∇: Levi-Civita (compatible with $\langle\,,\rangle$)

$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$

torsion-free: $\Gamma^k_{ij} = \Gamma^k_{ji}$

Γ not tensorial!

Let $a = w$, $b = c = v$:

$$\underbrace{\mathcal{L}_w \langle v, v\rangle}_{w\langle v, v\rangle} = \langle \nabla_w v, v\rangle + \langle v, \nabla_w v\rangle = 2\langle \nabla_w v, v\rangle$$

$$\Rightarrow \quad \langle \nabla_w v, v\rangle = \frac{1}{2}\, d\langle v, v\rangle(w)$$

Subsequently we have

$$(*) \quad \mathcal{L}_v(w, v) = \langle \nabla_v w, v\rangle + \langle w, \nabla_v v\rangle$$

$$= \langle \nabla_w v, v\rangle + \langle w, \nabla_v v\rangle$$

$$= \frac{1}{2}\, d\langle v, v\rangle(w) + \langle w, \nabla_v v\rangle$$

(use $[w, v] = 0$ + torsion-free character of ∇

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$)

Now $\mathcal{L}_\xi (v^b(w)) = (\mathcal{L}_\xi v^b)(w) + v^b \underset{\underset{[\xi, w]}{\shortparallel}}{\mathcal{L}_\xi w}$

if $\xi = v$, we get

$$\mathcal{L}_v (v^b(w)) = (\mathcal{L}_v v^b)(w) + \underset{\underset{0}{\shortparallel}}{v^b([v,w])} = (\mathcal{L}_v v^b)(w)$$

Moreover

$$\mathcal{L}_v (v^b(w)) = \mathcal{L}_v(\langle v, w\rangle)$$ by the very definition of b (+)

$$\Rightarrow \mathcal{L}_v (v^b)(w) \overset{(*)}{=} \mathcal{L}_v \langle v, w \rangle$$
$$\underset{(+)}{=} (\nabla_v v)^b (w) + \frac{1}{2} d\langle v, v\rangle (w)$$

Now, at x, if $v(x) \neq 0$, then one easily constructs w commuting with v, with arbitrary $w(x)$. If $v(x) = 0$, there is nothing to prove. Eventually we have (★)

$$\mathcal{L}_v (v^b) = (\nabla_v v)^b + \frac{1}{2} d\langle v, v\rangle$$

Upshot: another form of Euler's equation

From: $$\frac{\partial v}{\partial t} = -\nabla_v v - \nabla p$$ we have "lowering indices" (i.e. by applying b)

$$\frac{\partial v^b}{\partial t} = -(\nabla_v v)^b - dp$$

Riemannian gradient $\nabla f = (df)^\#$, $(\nabla f)^i = g^{ij}\partial_j f$

$$\overset{(★)}{=} -\mathcal{L}_v(v^b) + \frac{1}{2} d\langle v, v\rangle - dp$$

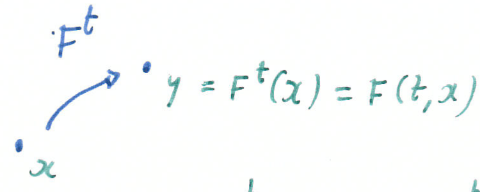
$\Rightarrow E'''$ $$\boxed{\frac{\partial v^b}{\partial t} = -\mathcal{L}_v(v^b) + d\left[\frac{1}{2}\langle v, v\rangle - p\right]}$$

Let us derive the vorticity form E^{IV}: set $w^b = dv^b$. Then (by $d^2 = 0$, $d\mathcal{L} = \mathcal{L}d$)

$\rightarrow E^{IV}$: $$\boxed{\frac{\partial w^b}{\partial t} + \mathcal{L}_v w^b = 0}$$

See Taylor PDE vol III

Some consequences of the Euler equation



$$\boxed{\frac{\partial \omega^b}{\partial t} + \mathcal{L}_u \omega^b = 0}$$

$$\omega^b(t)(x) \equiv \omega^b(t,x)$$

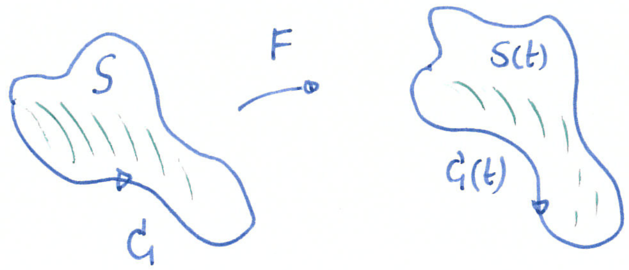
$$\Rightarrow \quad \omega^b(0) = (F^t)^* \omega^b(t) \qquad (\star)$$

$F^t \equiv F(t,x)$

(this comes from the very definition of $\mathcal{L}$)

(in 2-d we have $\omega(t,y) = \omega(0,x)$: conservation of vorticity)

Hence:



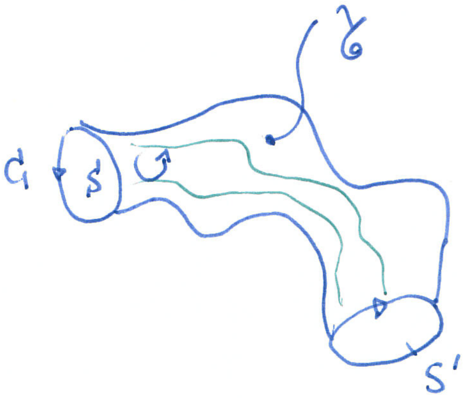
$$\int_{S(t)} \omega^b(t) = \int_S \omega^b(0)$$

$\Rightarrow$ ★ **Kelvin circulation theorem** (via Stokes)

$$\boxed{\int_{C(t)} v^b(t) = \int_C v^b(0)}$$

$\int_C v^b$ conserved

Then consider a flux tube (flow tube)



$$\int_\tau d\omega^b = \int_{\partial\tau} \omega^b = 0$$

$d\omega^b = d^2 v^b = 0$

$$\Rightarrow \int_S \omega^b = \int_{S'} \omega^b \quad \Rightarrow \text{(Stokes again)}$$

$$\boxed{\int_C v^b = \int_{C'} v^b}$$

C, C' determining a flux tube

↖ Strength of tube

(cf. Cartelli's theorem)

(★ Helmholtz theorem)

★ time independence of $H = \frac{1}{2}\int \|u(x,t)\|^2 d^3x$

$g = g(x,t) \in SDiff(\mathbb{R}^3)$

① $\quad u(x,t) = \frac{\partial g}{\partial t}(g_t^{-1}(x))$ $\qquad g(x,0) = x \quad (g_0 = id)$

g "rapidly approaching id at infinity"

$$\boxed{\int_{\mathbb{R}^3} \| u(x,t)\|^2 d^3x}$$

$$= \int_{\mathbb{R}^3} \left\| \frac{\partial g}{\partial t}(g_t^{-1}(x))\right\|^2 d^3x = \int_{\mathbb{R}^3} \left\| \frac{\partial g}{\partial t}(g_t^{-1}(x))\right\|^2 d^3(\underbrace{g_t^{-1}(x)}_{\equiv y})$$

$$= \int_{\mathbb{R}^3} \left\| \frac{\partial g}{\partial t}(x)\right\|^2 d^3x = \int_{\mathbb{R}^3} \left\| \frac{\partial g}{\partial t} g_0^{-1}(x)\right\|^2 d^3x$$

$$= \boxed{\int_{\mathbb{R}^3} \| u(x,0)\|^2 d^3x}$$

② Recall that $\quad \mathcal{L}_u f = i_u df + d(\underbrace{i_u f}_{=0}) = df(u) = \underbrace{u(f)}_{\equiv (u,\nabla f)}$

$\Downarrow$

$$(*) \quad \int_M \mathcal{L}_u f \, dv = \int_M (u, \nabla f)\, dv = -\int_M (\operatorname{div} u) f = 0 \quad \text{if } \operatorname{div} u = 0$$

↖ domain in $\mathbb{R}^3$, to be definite

$$\underbrace{\nabla(pu)}_{\operatorname{div}(pu)} = \nabla p \cdot u + p \operatorname{div} u$$

Now $\quad \langle \nabla_u X, X\rangle = \frac{1}{2} d\langle X, X\rangle(u) \qquad X \in \mathcal{X} \qquad (\nabla = \text{Levi-Civita})$

$\Rightarrow$ if $\operatorname{div} u = 0$

$$\int \langle \nabla_u X, X\rangle dv = \frac{1}{2}\int d\langle X,X\rangle(u)\, dv = \frac{1}{2}\int \mathcal{L}_u \langle X,X\rangle dv = 0$$

(by (*))

Then, by virtue of the Euler equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int \|u(t)\|^2 dv\right) = 2\cdot\frac{1}{2}\int \left\langle \frac{\partial u}{\partial t}, u\right\rangle dv$$

$$= -\underbrace{\int \langle \nabla_u u, u\rangle dv}_{=0} - \underbrace{\int \nabla p \cdot u}_{=0} = 0$$

★ Geodesics of a right-invariant metric on a Lie group G
(general discussion)

G : Lie group $\quad \mathfrak{g}$: Lie algebra of G

★ Canonical right-invariant 1-form ($\mathfrak{g}$-valued)

$$R = dg \cdot g^{-1}$$

$$d(g\,a)\cdot(g\,a)^{-1} = dg\;a\cdot a^{-1}g^{-1} = dg\cdot g^{-1}$$

constant

One has $\langle R, \partial_t\rangle = \partial_t g\cdot g^{-1}$ (example: Eulerian velocity)

The following Cartan structure equation holds:

(★) $$\boxed{dR - R\wedge R = 0}$$

left-invariance of a vector field
$(L_a)^* X_g = X_{ag}$
similarly for r. invariance

$\langle X, Y\rangle_g =$
$\langle (R_{g^{-1}})^* X, (R_{g^{-1}})^* Y\rangle_e$
right invariance of a metric $\langle , \rangle$

Pf: $dR = d\,(dg\,g^{-1}) = -\,dg\cdot dg^{-1}$

$$R\wedge R = dg\,g^{-1}\wedge dg\,g^{-1}$$

Now $\quad 0 = d(g g^{-1}) = dg\,g^{-1} + g\,dg^{-1}$

$$\Rightarrow\quad g\,dg^{-1} = -\,dg\cdot g^{-1}$$

$$\Rightarrow\quad dg^{-1} = -\,g^{-1}dg\,g^{-1}$$

Then
$dR = -dg\wedge dg^{-1} = dg\cdot g^{-1}\wedge dg\cdot g^{-1} = R\wedge R$

Let ∂_t, ∂_s two commuting v. fields. Then

$$(*) \quad d\mathcal{R}(\partial_t, \partial_s) = \partial_t \mathcal{R}(\partial_s) - \partial_s \mathcal{R}(\partial_t) - \mathcal{R}(\underbrace{[\partial_t, \partial_s]}_{=0})$$

Consider a variation $g = g(t,s)$ with fixed endpoints $g(0,s) = g(0)$, $g(1,s) = g(1)$

$g = g(t,0) = g(t)$ smooth curve in G

Define:

$$E(g) = \frac{1}{2}\int_0^1 \langle \mathcal{R}(\partial_t), \mathcal{R}(\partial_t)\rangle \, dt$$

right-invariant metric $(\dagger)$

$(\dagger)$ A scalar product on $\mathfrak{g} \cong T_e G$ can be promoted to a right (or left) invariant metric on G. An application of the Weyl trick shows that if G is compact, then it admits a biinvariant metric. In general, this is false.

Compute

$$\partial_s E(g)\Big|_{s=0} = \frac{1}{2}\cdot 2\int_0^1 \langle \partial_s \mathcal{R}(\partial_t), \mathcal{R}(\partial_t)\rangle \, dt$$

$$\overset{(*)}{=} \int_0^1 \langle \partial_t(\mathcal{R}(\partial_s)) - d\mathcal{R}(\partial_t, \partial_s), \mathcal{R}(\partial_t)\rangle \, dt$$

$$\underset{(\text{Cartan})}{=} \int_0^1 \langle \partial_t[\mathcal{R}(\partial_s)] - \underbrace{(\mathcal{R}\wedge\mathcal{R})(\partial_t, \partial_s)}_{=[\mathcal{R}(\partial_t), \mathcal{R}(\partial_t)]}, \mathcal{R}(\partial_t)\rangle \, dt$$

"free" this term

$$\underset{\text{integration by parts, fixed endpoints}}{=} -\int_0^1 \langle \underbrace{\mathcal{R}(\partial_s)}_{\text{this is arbitrary}}, \partial_t \mathcal{R}(\partial_t) + [\mathrm{ad}\, \mathcal{R}(\partial_t)]^T \mathcal{R}(\partial_t)\rangle \, dt$$

"coadjoint action" dual to the adjoint action ad

Upon looking for critical points, we find

$$\partial_t \mathcal{R}(\partial_t) + (\mathrm{ad}\,\mathcal{R}(\partial_t))^T \mathcal{R}(\partial_t) = 0 \qquad \mathcal{R}(\partial_t) = \frac{\partial g}{\partial t} g^{-1}$$

or, compactly $\mathcal{R}(\partial_t) \equiv u$

$$\boxed{u_t + \mathrm{ad}(u)^T u = 0} \quad \text{Euler equation}$$

$\rightsquigarrow$ recover rigid body ($G = SO(3)$, with left invariance) and perfect fluids ($G = \mathrm{SDiff}$ with caveat!!)

TOPICS IN SYMPLECTIC AND MULTISYMPLECTIC GEOMETRY

Ph.D. Course
Prof. M. Spera - UCSC - Brescia

Lecture XVI

★ Hamiltonian form of the Euler equation

(Arnold, Marsden - Weinstein, Penna - S.)

Start from Euler's equation in vorticity form

$$\boxed{\partial_t \omega = -[\,\omega, v\,]}$$

⚠ $[\ ] = -$ the usual one

Remember: $[a,b] = \text{curl}\,(a \times b)$ $\qquad a, b \in \mathfrak{g} = \text{sdiff}(\mathbb{R}^3)$

(✱) rapidly vanishing at infinity

Also

$$\text{div}\,(a \times b) = b \cdot \text{curl}\,a - a \cdot \text{curl}\,b$$

$$\Downarrow$$

$$\underbrace{\int \text{div}\,(a \times b)}_{=\,0} = \int b \cdot \text{curl}\,a - a \cdot \text{curl}\,b$$

(by (✱))

$$\int b \cdot \text{curl}\,a = \int a \cdot \text{curl}\,b$$

- Hamiltonian approach to Euler's equation
- Multisymplectic approach

Take then

$b = \text{curl}\,B$ ← B can be chosen to be divergence-free
$w = \text{curl}\,V$

$$\lambda_b(V) := \underbrace{\langle V, b\rangle}_{=\int v\cdot b} = \langle V, \text{curl}\,B\rangle = \langle \text{curl}\,V, B\rangle = \underbrace{\langle w, B\rangle}_{=\int w \cdot B}$$

$b \in \mathfrak{g}$
$V \in \mathfrak{g}$
velocity field

$$\boxed{\Lambda = \{\lambda_b\}_{b\in\mathfrak{g}} \equiv \text{Rasetti-Regge current algebra}}$$

$\Lambda \;\underset{\Downarrow}{\mathfrak{g}^*}$

Notice $\mathfrak{g} \subseteq \mathfrak{g}^*$ in this case

introduced by Rasetti & Regge (1975) in quantum vortex theory

★ If $w = \delta_\Gamma$

δ-like vorticity, concentrated on a filament

$$\lambda_b(\underset{v_\Gamma}{v}) = \int_\Gamma B \quad \text{the circulations}$$

One recovers the symplectic (Kähler) structure on the space of (mildly singular) knots of Brylinski (see also R-S)

★ **Theorem**

(i) $\Lambda = \{\lambda_b\}_{b \in \mathfrak{g}}$ is a Lie algebra → the Hamiltonian algebra of $\mathcal{O}_w$

(ii) $\boxed{\partial_t \lambda_b} = -\{H, \lambda_b\} = \boxed{\{\lambda_b, H\}}$ ← Kutnetsov-Mikhailov Poisson bracket (ii)' ≡ KKS - Poisson bracket ⇝ hydrodynamical bracket

$\partial_t \lambda_b = \dot{\lambda}_b$

(iii) $\dot{Q} = 0$ Helicity conservation (Moffatt)

$$\boxed{Q = \langle v, w \rangle = \int v \cdot w}$$

Helicity (Chern-Simons action)

Pf. Ad (i) : direct check

Ad (ii) $\{E, F\}(v) = \langle v\, [\frac{\delta E}{\delta v}, \frac{\delta F}{\delta v}] \rangle$ ← KM - Poisson brackets

$\frac{\delta E}{\delta v} = \operatorname{curl} \frac{\delta E}{\delta w}$

$= \langle w, \operatorname{curl} \frac{\delta E}{\delta w} \times \operatorname{curl} \frac{\delta F}{\delta w} \rangle$

formal calculations

$\operatorname{curl} \frac{\delta E}{\delta w} = \operatorname{curl} \frac{\delta E}{\delta \operatorname{curl} v} = \cancel{\operatorname{curl}} \frac{\delta E}{\cancel{\operatorname{curl}} \delta v} = \frac{\delta E}{\delta v}$

Then $\boxed{\{H, \lambda_b\}(v)} = \langle v, [v, b] \rangle$

$v = \frac{\delta H}{\delta v}$, $b = \frac{\delta \lambda_b}{\delta v} = \operatorname{curl} B$ ← "vector potential" for b

$= \langle w, v \times b \rangle = \langle b, w \times v \rangle$

$= \langle \operatorname{curl} B, \bar{w} \times \bar{v} \rangle = \langle B, [w, v] \rangle$

$[w, v] = -\partial_t w$ ★ Euler

$= \langle B, -\partial_t w \rangle$

$= \langle B, -\partial_t \operatorname{curl} v \rangle = \langle B, \operatorname{curl}(-\partial_t v) \rangle$

$= \langle b, -\partial_t v \rangle = -\partial_t \langle b, v \rangle = \boxed{-\dot{\lambda}_b}$

$$\boxed{\dot{\lambda}_b = \{\lambda_b, H\}}$$

★ Hamiltonian form of Euler's equation

(ii)' (★) Actually, $\{\, , \}_{KM}$ coincides with KKS - PB (the vorticity form of E expresses motion on a coadjoint orbit of $\mathfrak{g}$, labelled by $[v]$ or w, $\mathcal{O}_w$ ⇝

Ad (iii)

$$\boxed{\partial_t Q} = \partial_t \langle v, w\rangle = \langle \underbrace{\partial_t v}_{\lambda_W}, w\rangle + \langle v, \partial_t w\rangle$$

$$= -\{H, \lambda_W\} - \langle v, [w, v]\rangle$$

$$= -\langle v, [\underset{\substack{\| \\ v}}{\frac{\delta H}{\delta v}}, \underset{\substack{\| \\ w}}{\frac{\delta \lambda_W}{\delta v}}]\rangle - \langle v, [w, v]\rangle$$

$$= -\langle v, [v, w]\rangle + \langle v, [v, w]\rangle = 0$$

Recall
$\langle ad^*x \cdot f, a\rangle$
$= -\langle f, ad_x(a)\rangle$
$= -\langle f, [x, a]\rangle$
in our case $f \sim v$
$= -\int v\, \mathrm{curl}\, x \times a$
$= -\int w \cdot x \times a$
$= -\int x \cdot a \times w$
$= -\int a \cdot w \times x$
$= \int a\,(-w \times x)$

$$\boxed{\partial_t Q = 0}$$ □

recall $\eta^\#: f \longmapsto f + ad^*\eta$
$\langle (d\lambda_\xi)_f\, \eta^\#_f\rangle = (\eta^\# \lambda_\xi)(f) = \langle ad^*\eta \cdot f, \xi\rangle = -\langle f, ad\,\eta\, \xi\rangle$
$= -\langle f\, [\eta, \xi]\rangle = \langle f, [\xi, \eta]\rangle = \Omega_f(\xi^\#_f, \eta^\#_f)$
$\{\lambda_\xi, \lambda_\eta\} = \lambda_{[\xi,\eta]}$
not relevant

Let us comment on (◈)

$$ad_x(v) = [x, v] \qquad ad^*_x(v) = -w \times x - \nabla(v \cdot x)$$

adjoint action — coadjoint action $=: x^\#_v \;(= x^\#_{[v]})$

KKS: $\{\lambda_a, \lambda_b\}([v]) = \Omega_{[v]}(a^\#_{[v]}, b^\#_{[v]}) = \langle v, [a, b]\rangle = \langle w, a \times b\rangle$

$\| \; ad^*_{[v]}(a)$

recall
$\mathrm{div}(pb) = \nabla p \cdot b + p\,\mathrm{div}\, b$
If $\mathrm{div}\, b = 0$, then $\nabla p \cdot b = \mathrm{div}(pb)$

Notice this:
$dH = i_v \Omega$
$d\lambda_b = i_{b^\#} \Omega$

$$\Omega_{[v]}(v^\#_{[v]}, b^\#_{[v]}) = \langle v, [v, b]\rangle = \langle w, v \times b\rangle$$

agree

disappears

$$dH(b^\#) = \langle -w \times b - \nabla(\;), v\rangle = -\langle w \times b, v\rangle$$
$$= \langle v \times b, w\rangle$$

$\| \; \frac{1}{2}\langle v, v\rangle$ — $ad^*_{[v]} b$

$(b^\# H)(v)$

KM: $\{\lambda_a, \lambda_b\}([v]) = \langle v\, [\frac{\delta \lambda_a}{\delta v}, \frac{\delta \lambda_b}{\delta v}]\rangle$
$= \langle v, [a, b]\rangle = \langle w, a \times b\rangle$

$\frac{\delta \lambda_a}{\delta v} = a$

(Same argument with $a^\#$ replacing $v^\#$, λ_a replacing H)

Ideal fluids : multisymplectic picture

$(\mathcal{B}, G)$ $\quad \mathcal{B}$ n-dimensional, compact, oriented (with smooth boundary $\partial\mathcal{B}$), riemannian

(M, g) $\quad N$-dim, compact, oriented, riemannian

$\ast$ Fluid case : $\quad \mathcal{B} = M$ (reference fluid configuration)

$X = \underset{\ni x^i}{\mathcal{B}} \times \underset{\ni t = x^0}{\mathbb{R}}$ $\qquad x^\mu = (x^i, x^0) = (x, t)$

$Y = X \times M \to X$

$(\underbrace{x, t}_{X}, \underset{M}{y}) \mapsto \underset{X}{(x, t)}$

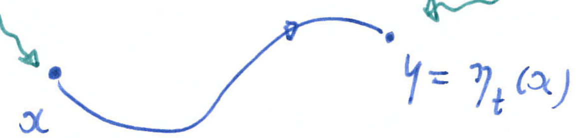
fluid flow : $\quad \eta_t : M \to M \qquad \eta_0 = id$

$$\boxed{\eta_t(\cdot) = \text{fluid configuration at time } t}$$

$x \in M$ original position $\qquad y := \eta_t(x)$ position at t

material point

Spatial point



material velocity $\rightsquigarrow$ $V(x,t) = \dfrac{\partial \eta_t(x)}{\partial t} \equiv \dot{\eta}$

$u = u(y,t)$ $\leftsquigarrow$ spatial velocity

$$\boxed{u(y,t) := V(x(y), t) = V(\eta_t^{-1}(y), t)}$$

$$\boxed{u = V \circ \eta_t^{-1} = \dot{\eta} \cdot \eta_t^{-1}}$$

☆ Incompressible continuum mechanics
(via Lagrange multipliers)

$$\pi_{XE} : E \xrightarrow{M\times\mathbb{R}} X \qquad (x,t,\ y,\ \lambda) \longmapsto (x,t)$$

$$\begin{array}{cccc} (x,t) & y & \lambda & (x,t) \\ \cap & \cap & \cap & \cap \\ X & M & \mathbb{R} & X \end{array}$$

section of the trivial bundle $X\times\mathbb{R} \to X \equiv \mathcal{V} \to X$

$\lambda = \lambda(x,t)$

multiplier

☆ phase space : first jet bundle $J^1(E)$, with coordinates

$$\gamma = (\ x^\mu,\ y^a,\ \lambda,\ v^a_\mu,\ \beta_\mu)$$

☆ first jet extension of $\bar\phi = (\phi,\lambda)$, section of E

$$\boxed{j^1\bar\phi = (x^\mu,\ y^a,\ \lambda,\ \partial_\mu\phi^a,\ \partial_\mu\lambda)}$$

$(J^1E)^*$: affine dual bundle $\equiv Z$ vertically invariant subbundle of $\Lambda = \Lambda^{n+1}E$

$$Z \ni z = \Pi\, d^{n+1}x + p_a{}^\mu\, dy^a \wedge d^n x_\mu + \pi^\mu d\lambda \wedge d^n x_\mu$$

" $\Theta = z$ " (tautological form)

$$\boxed{\Theta = (p^\mu_a\, dy^a + \pi^\mu d\lambda) \wedge d^n x_\mu + \Pi\, d^{n+1}x}$$

$$\boxed{\Omega = -d\Theta}$$

Determine the "primary constraint manifold" $\mathcal{E}$ in our context

$$i_{\mathcal{E}} : \mathcal{E} \hookrightarrow J^1E^* \qquad i_{\mathcal{E}}\Omega = \Omega_{\mathcal{E}} \text{ will be degenerate}$$

* incompressibility $\equiv$ pointwise constraint defined on J^1Y

$Y: M\times X \to X$

$$\gamma \equiv (x^\mu, y^a, v^a_\mu) \in J^1 Y$$

$$\bar{\gamma} \equiv (x^\mu, y^a, \lambda, v^a_\mu, \beta_\mu)$$

ϕ^a $\quad \partial_\mu \phi^a$ $\quad \partial_\mu \lambda$

$$\boxed{\Phi(\gamma) = 0} \qquad \Phi : J^1 Y \to \mathbb{R}$$

$$\gamma \mapsto J(\gamma) - 1$$

$$\parallel$$

$$\det[v] \sqrt{\frac{\det g(y)}{\det G(x)}}$$

$v = v^a_i \quad \partial_i \phi^a$

spatial indices (so $\det[v]$ is well-defined)

* Lagrange multiplier

$$\lambda(x) = \sqrt{\det G} \cdot p(x) \qquad \lambda : X \to \mathbb{R}$$

material pressure

Lagrangian density

$$\mathcal{L} : J^1 E \to \Lambda^{n+1} X$$

$$\boxed{\mathcal{L}_\Phi(\bar{\gamma})} = \boxed{(\underbrace{L(\gamma) + \lambda\, \Phi(\gamma)}_{L_\Phi})\, d^{n+1}x}$$

$$L(\gamma) = K - P$$

λ is cyclic: $\quad \pi^\mu = \dfrac{\partial \mathcal{L}_\Phi}{\partial \beta_\mu} = 0 \qquad$ (conjugate momenta)

$\mathbb{F}\mathcal{L}_\Phi : J^1 E \to J^1 E^*$ degenerate, get a primary constraint $\pi^\mu = 0$

★★★ Multisymplectic Euler-Lagrange equations

$$\boxed{\frac{\partial L_\Phi}{\partial y^a}(j^1\bar\phi) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L_\Phi}{\partial v^a_\mu}(j^1\bar\phi)\right) = 0}$$

E-L for λ recovers the constraint $\Phi = 0$

$$\boxed{\frac{\partial L_\Phi}{\partial \lambda} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L_\Phi}{\partial \beta_\mu}(j^1\bar\phi)\right) = \Phi(j^1\phi) = 0}$$

to be solved together

working with the general Lagrangian

$$L = \mathbb{K} - \mathbb{P} = \sqrt{\det G}\,\rho(x)\,g_{ab}\,v^a_0 v^b_0 - \sqrt{\det G}\,\rho\,\bar W(x, G(x), g(y), v^a_j)$$

kinetic energy — potential energy — stored energy function

one gets (including the material pressure term, coming from the multiplier)

$$\rho\, g_{ab}\left(\frac{D_g\dot\phi}{Dt}\right)^b - \frac{1}{\sqrt{\det G}}\frac{\partial}{\partial x^k}\left(\rho\frac{\partial W}{\partial v^a_k}(j^1\phi)\sqrt{\det G}\right)$$

$$= -\rho\frac{\partial W}{\partial g_{bc}}\frac{\partial g^{bc}}{\partial y^a}(j^1\phi) - \frac{\partial P}{\partial x^k}(v^{-1})^k_a\, J(j^1\phi)$$

In our special case ($\rho = 1$, $g = G$ = Euclidean metric, $W = 0$)

$$\rightsquigarrow \quad \frac{d\dot\phi_a}{dt} = -\frac{\partial P}{\partial x^k}(v^{-1})^k_a\,\underbrace{J(j^1\phi)}_{=1}$$

$(v = \partial\phi)$

and, upon setting $\phi(x,t) = \eta_t(x)$ we recover the classical Euler equation for perfect fluids, see next page

Comparison of standard & multisymplectic approaches to Euler's equation

$$\mathcal{G} = \mathcal{G}_E(y) = \mathcal{G}_L(x)$$

generic **tensor** — Eulerian viewpoint — Lagrangian viewpoint

$y = \eta_t(x)$

$$\phi(x,t) = \eta_t(x)$$

$\phi(x,t) = y$: position of "particle" x at time t — spatial coordinate

$\eta_t(x)$: reference configuration of fluid — material coordinate

$$\frac{d\mathcal{G}_E}{dt}(y) = \frac{\partial \mathcal{G}_E}{\partial t}(y) + u(y)\cdot\nabla \mathcal{G}_E$$

Lagrangian derivative — Eulerian derivative

$\mathcal{E}$	$\mathcal{L}$
$u(y)$ =	$\bar{V}(x) = \partial_t \eta_t(x) \equiv \dot{\eta}$

$$u \circ \eta = \dot{\eta}$$

$$u = \dot{\eta}\,\eta^{-1}$$

in particular $\mathcal{G}_E = u$

spatial pressure $p = P\circ\phi^{-1}$ — material pressure

Euler equation ① $\boxed{\partial_t u + u\cdot\nabla u = -\nabla p}$ (evaluated at y)

★ Multisymplectic E-L

with Lagrange multiplier λ... material pressure

(inviscid incompressible fluid — standard metrics throughout)

≡ perfect

$\rho = \text{constant} = 1$ $\qquad \dot{\phi} = \partial_t\phi$

② $$\boxed{\frac{d}{dt}\dot{\phi} = -\frac{\partial P}{\partial x^k}\left((v^{-1})^k{}_a\right)\underbrace{J(j^1\phi)}_{=1}}$$

$v = \partial\phi$

incompressibility constraint

$y = \phi(x)$ $\quad x = \phi^{-1}(y)$

$$\frac{\partial p}{\partial y} = \frac{\partial(P\circ\phi^{-1})}{\partial y} = \frac{\partial P}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial y}}_{\partial\phi^{-1}}$$

chain rule

⇝ (working at y)

★ we get ① = ②

($\frac{d}{dt}$ = Lagrangian derivative $= \frac{\partial}{\partial t} + u\cdot\nabla$)

Generalized Kelvin theorems

(After Ryvkin - Wurzbacher - Zambon) 2016

Let M be a (smooth) manifold, $v \in \mathfrak{X}(M)$

Σ: compact, oriented d-manifold (a "membrane" evolving via v) with boundary

$\sigma_0 : \Sigma \to M$ smooth map

ϕ_t : flow of v ; $\sigma_t = \phi_t \circ \sigma$

Theorem (generalized Kelvin)

In the above notation, let $\alpha \in \Omega^d(M)$. Then

$$\boxed{\int_\Sigma (\sigma_t)^* \alpha \quad \text{is independent of } t}$$

provided one of the following conditions holds

(i) α is strictly conserved by v $\qquad \mathcal{L}_v \alpha = 0$

(ii) α is globally conserved by v and Σ has no boundary $\partial\Sigma = \phi$ $\qquad \mathcal{L}_v \alpha = d\beta$ ($\mathcal{L}_v\alpha$ exact)

(iii) α is locally conserved by v $\qquad \Leftrightarrow \quad d(\mathcal{L}_v \alpha) = 0$ ($\mathcal{L}_v\alpha$ closed)

and $\exists$ N, compact, oriented, $\partial N = \Sigma$ and

$\tilde{\sigma}_0 : N \to M$ with $\tilde{\sigma}_0 \big|_{\underset{\Sigma}{\partial N}} = \sigma_0$

Remark: In view of compactness of Σ, we may assume ϕ_t defined on $(-\varepsilon, \varepsilon) \times \sigma_0(\Sigma) \subset \mathbb{R} \times M$, $\varepsilon = \varepsilon(\sigma_0) > 0$ ($\varepsilon = \varepsilon(\tilde{\sigma}_0)$ in case iii)

If ① $\dim M = d$, or ② $d\alpha = 0$,

Then α is globally conserved

① $\mathcal{L}_v \alpha = d\, i_v \alpha + i_v \underset{m+1}{d\alpha} = d(i_v \alpha)$

② Same calculation

Proof: we just check (ii)

Start from

$$(\sigma_0^*)\ \frac{d}{dt}(\phi_t^*\alpha) = \phi_t^* \mathcal{L}_v \alpha \quad (\sigma_0^*)$$

$\sigma_0^*\phi_t^* = (\phi_t\circ\sigma_0)^* = \sigma_t^*$

Apply $(\sigma_0)^*$:

$$\frac{d}{dt}(\sigma_t^*\alpha) = \sigma_t^* \mathcal{L}_v \alpha$$

Integrate (use compactness of Σ)

$$\boxed{\frac{d}{dt}\int_\Sigma \sigma_t^*\alpha} = \int_\Sigma \sigma_t^*(\mathcal{L}_v\alpha) = \int_\Sigma \sigma_t^* d\gamma = \int_\Sigma d(\sigma_t^*\gamma) = 0$$

$\mathcal{L}_v\alpha = d\gamma$

Stokes $+\ \partial\Sigma = 0$

□

If α is locally conserved, and $\partial\Sigma=\phi$, one also has a map

$$F_t : [\Sigma, M] \to \mathbb{R}, \quad \boxed{[\sigma_0] \longmapsto \int_\Sigma(\sigma_t^*)\alpha - \int_\Sigma(\sigma_0)^*\alpha}$$

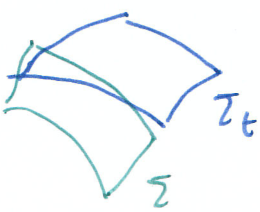
homotopy classes of smooth maps $f:\Sigma\to M$

$\sigma_t := \phi_t\circ\sigma_0 : \Sigma \to M$

Also:

$$\boxed{F_t[\sigma_0] = t\cdot \underbrace{\int_\Sigma(\sigma_0)^*(\mathcal{L}_v\alpha)}_{\text{III}\ c[\sigma_0]}}$$

(linear dependence on t)



Indeed

$$\boxed{\int_\Sigma (\sigma_t^*)\alpha - \int_\Sigma (\sigma_0)^*\alpha} = \int_0^t \left[\frac{d}{ds}\int_\Sigma (\sigma_s^*)\alpha\right] ds$$

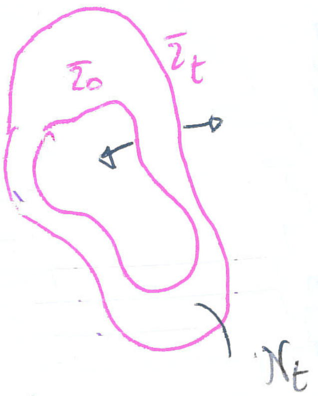
$$= \int_0^t \Big[\underbrace{\int_\Sigma \sigma_s^*(\mathcal{L}_v\alpha)}\Big]\, ds$$

Now $d(\mathcal{L}_v\alpha) = 0$

only depends on the homotopy class of σ_s^*, i.e., σ_0. In general

$\sigma_t \sim \sigma_0$

$$\int_\Sigma \underbrace{\sigma_t^*(\mathcal{L}_v\alpha)}_{\omega_t} - \int_\Sigma \underbrace{(\sigma_0)^*(\mathcal{L}_v\alpha)}_{\omega_0}$$



$$= \int_{\Sigma_t} \omega_0 - \int_{\Sigma_0} \omega$$

$$= \int_{\partial N_t} \omega_0 = \int_{N_t} \overset{0}{\overset{\shortparallel}{d\omega_0}}$$

$$= 0$$

$$\int_0^t \Big(\int_\Sigma \sigma_0^*(\mathcal{L}_v\alpha)\Big)dt$$

$$\shortparallel$$

$$t \cdot \underbrace{\int_\Sigma (\sigma_0^* \mathcal{L}_v\alpha)}$$

$$\equiv$$

$$c[\sigma_0]$$

$$\shortparallel$$

$$\boxed{t \cdot c[\sigma_0]}$$

* Kelvin's circulation theorem extended

(variant of the previous theorem)

$$\alpha^t \in \Omega^d(M) \quad , \quad \mathcal{L}_{v^t} \alpha^t + \frac{d\alpha^t}{dt} \text{ exact}, \quad \partial\Sigma = \emptyset$$

$$\Rightarrow \boxed{\frac{d}{dt}\left[\int_\Sigma (\sigma_t)^* \alpha^t\right] = 0}$$

Ordinary Kelvin's theorem is a special case of

$v^t = (v^t)^i \partial_i$ time dependent v.f. on $\mathbb{R}^3$

$\leadsto$ via the standard metric get $(v^t)^b = (v^t)_i dx^i$ (a 1-form), $(v^t)_i = (v^t)^i$

and $\quad d(v^t)^b = \frac{\partial v_i}{\partial x^j} dx^j \wedge dx^i$

Then: $\quad i_v dv^b = \frac{\partial v_i}{\partial x^j} dx^j(v) dx^i - \frac{\partial v_i}{\partial x^j} dx^i(v) dx^j$

$$= \frac{\partial v_i}{\partial x^j} v^j dx^i - \frac{\partial v_i}{\partial x^j} v^i dx^j$$

$$= \left(\frac{\partial v_i}{\partial x^j} v^j - \frac{\partial v_j}{\partial x^i} v_j\right) dx^i$$

$$= \frac{\partial v_i}{\partial x^j} v^j . dx^i - \frac{1}{2} d(v_j v^j)$$

From $\mathcal{L}_v v^b = i_v dv^b + d(i_v v^b) = i_v dv^b + d\underbrace{v^b(v)}_{\Sigma v_i^2}$

we get

$$\mathcal{L}_{v^t}\alpha^t + \frac{d\alpha^t}{dt} \text{ exact} \Leftrightarrow (*) \quad \frac{\partial v_i}{\partial x^j} v^j dx^i + \frac{dv_i^t}{dt} dx^i \text{ is exact}$$

Then $(\blacklozenge)$ $\int_\Sigma (\sigma_t)^* \alpha^t$ is independent of t

But $(*)$ can be reformulated as $(v.\nabla)v + \frac{\partial v}{\partial t} = -\nabla p$

i.e. it is Euler's equation, and $(\blacklozenge)$

becomes the standard Kelvin theorem $\square$

Multisymplectic manifolds

(à la Rogers - Wurzbacher - Zambon)

- multisymplectic manifolds
- L_∞-algebras

n-plectic / multisymplectic manifold

(M, ω) $\quad \omega \in Z^{n+1}(M)$ $\quad$ (i.e. $d\omega = 0$) $\}$ pre n-plectic manifold

If, in addition, the map

$$T_pM \longrightarrow \Omega^n T_p^*M$$
$$v \longmapsto i_v \omega_p$$

is <u>injective</u> $\forall\, p \in M$

we call (M, ω) n-plectic / multisymplectic

$n=1$ yields back <u>symplectic</u> geometry

Let (M, ω) be a pre n-plectic manifold

$\alpha \in \Omega^{n-1}(M)$ is called Hamiltonian if $\exists\; v_\alpha \in \mathfrak{X}(M)$ such that

$$\boxed{d\alpha = -\, i_{v_\alpha}\omega}$$

v_α : Hamiltonian v. field for α (in the n-plectic case v_α is unique)

$\Omega^{n-1}_{Ham}(M)$: Hamiltonian $(n-1)$-forms

Obviously $\quad \boxed{\mathcal{L}_{v_\alpha}\omega} \overset{\text{Cartan}}{=} i_{v_\alpha}\overset{\overset{0}{\shortparallel}}{d\omega} + d\, i_{v_\alpha}\omega$

$$= -d(d\alpha) = \boxed{0}$$

Lie n-algebra of observables

$$L_\infty(M,\omega) = (L, \{l_k\})$$

graded vector space:

$$\boxed{L_i = \begin{cases} \Omega^{n-1}_{Ham}(M) & i=0 \\ \Omega^{n-1-i}(M) & 0 < i \le n-1 \end{cases} \quad (*)}$$

$$\{l_k : L^{\otimes k} \to L, \quad 1 \le k \le n+1\}$$

$$l_1(\alpha) = d\alpha \qquad \deg\alpha > 0$$

$$l_1(\alpha) = 0 \qquad \deg\alpha = 0$$

in the sense of $(*)$

and, for $k > 1$

$$l_k(\alpha_1, \dots, \alpha_k) = \begin{cases} 0 & \deg(\alpha_1 \otimes \dots \alpha_k) > 0 \\ \varsigma(k)\, \underbrace{i(v_{\alpha_1} \wedge v_{\alpha_2} \wedge \dots \wedge v_{\alpha_k})}_{i_{v_{\alpha_k}} i_{v_{\alpha_{k-1}}} \dots i_{v_{\alpha_1}}}\omega & \text{if } \deg(\alpha_1 \otimes \dots \alpha_k) = 0 \end{cases}$$

$$\varsigma(k) = -(-1)^{k(k+1)/2}$$

contraction

M manifold, $v \in \mathfrak{X}(M)$
$\alpha \in \Lambda^\bullet(M)$ is called

$C_{loc}(v) \leftarrow$ (a) locally conserved: $\mathcal{L}_v\alpha$ closed $\quad (d(\mathcal{L}_v\alpha) = 0)$

$\cup$

$C(v) \leftarrow$ (b) globally conserved $\mathcal{L}_v\alpha$ exact $\quad (\mathcal{L}_v\alpha = d\beta)$

$\cup$

$C_{str}(v) \leftarrow$ (c) strictly conserved $\mathcal{L}_v\alpha = 0$

Also $Z(M) \subset C(v)$ $\qquad \mathcal{L}_v\alpha = i_v d\alpha + d i_v\alpha = d(i_v\alpha)$

$d\, C_{loc}(v) \subset C_{str}(v)$ $\qquad \mathcal{L}_v d\beta = d\mathcal{L}_v\beta = 0$

$\beta \in C_{loc}(v)$

Additional results for (M,ω) pre n-plectic
& v preserving ω $(\mathcal{L}_v\omega = 0)$

(i) $\alpha \in \Omega^{n-1}_{Ham}(M)$ is **locally conserved** by v
$\iff$ $i_{[v_\alpha, v]}\omega = 0$ for some (eq. every) Hamiltonian v. field v_α

$d\alpha = -i_{v_\alpha}\omega$

Recall
$\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]}$

α locally conserved

$\boxed{d(\mathcal{L}_v\alpha) = 0}$ $\iff$ $\mathcal{L}_v d\alpha = 0$

$\iff$ $\mathcal{L}_v i_{v_\alpha}\omega = 0$ $\iff$ $i_{v_\alpha}\underbrace{\mathcal{L}_v\omega}_{=0} - i_{[v,v_\alpha]}\omega = 0$

$\iff$ $\boxed{i_{[v,v_\alpha]}\omega = 0}$

If $v = v_H$ is Hamiltonian for $H \in \Omega^{n-1}_{Ham}(M)$, then

(ii) $\alpha \in \Omega^{n-1}_{Ham}(M)$ l. cons. by v_H $\iff$ $\mathcal{L}_{v_\alpha}H$ is **closed** for some (eq. every) Hamiltonian v.f. v_α for α

$dH = -i_v\omega$ Consider $\mathcal{L}_{v_\alpha}H$. Then

$0 = d\mathcal{L}_{v_\alpha}H = \mathcal{L}_{v_\alpha}dH = -\mathcal{L}_{v_\alpha}i_v\omega$

But $\left[0 = \mathcal{L}_{v_\alpha}i_v\omega = i_v\underbrace{\mathcal{L}_{v_\alpha}\omega}_{=0} + i_{[v_\alpha,v]}\omega\right]$

$\iff$ α is **locally conserved** by $v = v_H$

(iii) $\alpha \in \Omega^{n-1}_{Ham}(M)$ is globally conserved $\Leftrightarrow$ $\mathcal{L}_{v_\alpha} H$ is exact for some (eq. every) Hamiltonian v.f. v_α

$v = v_H$

let $\mathcal{L}_{v_\alpha} H = d\beta$

compute$^{(\diamond)}$ $\boxed{\mathcal{L}_v \alpha} = i_v d\alpha + d\, i_v \alpha =$

$= i_v(-i_{v_\alpha}\omega) + d(i_v\alpha) = \boxed{\underbrace{i_{v_\alpha} i_v \omega}_{(*)} + d(i_v\alpha)}$

$= -i_{v_\alpha} dH + d(i_v\alpha)$

$= d\, i_{v_\alpha} H - \mathcal{L}_{v_\alpha} H + d\, i_v\alpha =$

$d\, i_{v_\alpha} H - d\beta + d\, i_v\alpha$

$(i_X i_Y \omega)(z_1 - z_n)$

$= i_X(i_Y\omega)(z_1 - z_n)$

$= (i_Y\omega)(X\, z_1 - z_n)$

$= \omega(Y, X, z_1 - z_n)$

$\boxed{\mathcal{L}_{v_\alpha} H = i_{v_\alpha} dH + d\, i_{v_\alpha} H}$ $= \boxed{d[\, i_{v_\alpha} H - \beta + i_v\alpha\,]}$

we proved $(\Leftarrow)$, but the argument is reversible, whence the conclusion follows.

record (use $\diamond$)

$\mathcal{L}_{v_H}\alpha = \underbrace{i_{v_H}[\overset{d\alpha}{-i_{v_\alpha}\omega}]}_{+(*)} + d(i_{v_H}\alpha)$

(iv) $H \in C(v_H)$

This means $\mathcal{L}_{v_H} H$ exact.

compute: $\boxed{\mathcal{L}_{v_H} H} = d\, i_{v_H} H + i_{v_H} dH =$

$= d(i_{v_H} H) + \underbrace{i_{v_H}[-i_{v_H}\omega]}_{\overset{\shortparallel}{0}} = \boxed{d(i_{v_H} H)}$

Compute $l_2(\alpha, H) = \underbrace{-(-1)^{\binom{3}{2}}}_{+1} i(v_\alpha \wedge v_H)\omega = i_{v_H} i_{v_\alpha}\omega = -(*)$

$\Rightarrow$ $\mathcal{L}_{v_\alpha} H$ is closed (exact) $\Leftrightarrow$ $l_2(\alpha, H)$ is closed (exact)

☆ Important: $\mathcal{L}_{V_H} H \neq 0$ in general

$M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$, $H = x\,dy + z\,dz$

V_H: $\quad i_{V_H} \omega = -dH \qquad\qquad dH = dx \wedge dy$

$$V_H = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + \gamma \frac{\partial}{\partial z}$$

$$i_{V_H}\omega = \alpha\, dy \wedge dz - \beta\, dx \wedge dz + \gamma\, dx \wedge dy = -dx \wedge dy$$

$$\Rightarrow \quad \alpha = \beta = 0, \quad \gamma = -1 \quad \Rightarrow \quad V_H = -\frac{\partial}{\partial z}$$

$$i_{V_H} H = -z \qquad \mathcal{L}_{V_H} H = \underbrace{i_{V_H} \overset{0}{\overset{\|}{dH}}}_{0} + d\, i_{V_H} H = -dz$$

(exact, as it should be...)

☆ Symplectic case (1-plectic) $\quad$ [f globally conserved: $\mathcal{L}_{V_H} f$ exact, that is "0"]

$$\mathcal{L}_{V_H} f = i_{V_H} df = V_H(f)$$

$\mathcal{L}_{V_H} f$ closed means $d[df(V_H)] = 0$

$\Rightarrow \quad V_H(f)$ constant

$\Downarrow$ we have $\{f, H\}$ constant

in the globally conserved case:

$V_H(f) = 0$ i.e. $\{f, H\} = 0$

i.e. f strictly conserved

Example: (symplectic case) $\quad$ f locally conserved $\not\Rightarrow$ f globally conserved

$\mathbb{R}^2$

$\omega = dp \wedge dq \qquad f = q \qquad\qquad H = p$

$$\{q, p\} = 1 \neq 0$$

★ Algebraic structure of conserved quantities

new conserved quantities from old ones $\quad v \in \mathfrak{X}(M)$

$C_{str}(v)$ is a graded subalgebra of $\Omega^\bullet(M)$

but $C(v)$ & $C_{loc}(v)$ are not closed under $\wedge$

Example: $M = \mathbb{R}^3 \quad \omega = dx \wedge dy \wedge dz \quad H = -x\,dy$

$dH = -dx \wedge dy \quad v_H = \frac{\partial}{\partial z} \quad \alpha := z\,dx \quad \beta = z\,dy$

$d\alpha = dz \wedge dx \quad d\beta = dz \wedge dy$

$$\mathcal{L}_{v_H} \alpha = i_{v_H} d\alpha + d \underbrace{i_{v_H} \alpha}_{=0} = i_{v_H} d\alpha = dx$$

α globally conserved

$$\mathcal{L}_{v_H} \beta = i_{v_H} d\beta + d \underbrace{i_{v_H} \beta}_{=0} = i_{v_H} d\beta = dy$$

β globally conserved

However $\alpha \wedge \beta = z^2 dx \wedge dy$

$$\mathcal{L}_{v_H}(\alpha \wedge \beta) = d \underbrace{i_{v_H}(z^2 dx \wedge dy)}_{=0} + i_{v_H} d(z^2 dx \wedge dy)$$

$$= \underset{= \frac{\partial}{\partial z}}{i_{v_H}}(2z\, dx \wedge dy \wedge dz) = 2z\, dx \wedge dy$$

not even closed!

Define $\mathcal{A}(v) := \{ \beta \in \Omega(M) \,/\, d\beta = 0,\ \mathcal{L}_v \beta = 0 \} \subset C_{str}(v)$ (clear)

Then: $C(v)$ and $C_{loc}(v)$ are graded modules over $\mathcal{A}(v)$

Proof for $C(v)$ $\quad \alpha \in C(v)$: $\mathcal{L}_v \alpha = d\gamma$

$\beta \in \mathcal{A}(v)$

$$\boxed{\mathcal{L}_v(\alpha \wedge \beta)} = \mathcal{L}_v \alpha \wedge \beta + \alpha \wedge \underbrace{\mathcal{L}_v \beta}_{=0} = \mathcal{L}_v \alpha \wedge \beta = d\gamma \wedge \beta$$

$$= \boxed{d(\gamma \wedge \beta)} \quad \square$$

Thm: (M,ω) pre n-plectic ($d\omega=0$), $v\in\mathcal{X}(M)$, $\mathcal{L}_v\omega=0$

Then
$$\left.\begin{array}{l} L_\infty(M,\omega)\cap C_{loc}(v) \\ L_\infty(M,\omega)\cap C(v) \\ L_\infty(M,\omega)\cap C_{str}(v)\end{array}\right\}$$
are L_∞-subalgebras of $L_\infty(M,\omega)$

Also:
$$\mathcal{L}_v\left(l_k(\beta_1 - \beta_k)\right)=0$$
for $k\geq 1$ & $\beta_1\cdots\beta_k\in L_\infty(M,\omega)\cap C_{loc}(v)$

we prove the following:

brackets of l. c. quantities in $L_\infty(M,\omega)$ are strictly conserved

The only non trivial bracket on components different from $\Omega^{n-1}_{Ham}(M)$ is $l_1=d$. Application of $l_1=d$ to a locally conserved quantity yields a strictly conserved quantity: ($d\, C_{loc}\subset C_{str}$)

Now let $k\geq 2$ and take $\beta_1 - \beta_k\in\Omega^{n-1}_{Ham}(M)$ with $\mathcal{L}_v\beta_i$ closed $\forall i$

claim: $\mathcal{L}_v\big(\underbrace{l_k(\beta_1-\beta_k)}_{\pm i(v_{\beta_1}\wedge - v_{\beta_k})\omega}\big)=0 \qquad (*)$

$(*)$ is equivalent to $\overbrace{\mathcal{L}_v(i_{v_{\beta_k}} - i_{v_{\beta_1}}\omega)}^{(\blacklozenge)}=0$

Now: use $\boxed{\mathcal{L}_X i_Y - i_Y\mathcal{L}_X = i_{[X,Y]}}$ in conjunction with $\boxed{i_{[v,v_{\beta_i}]}\omega=0}$

we find (we can move $\mathcal{L}_v$ past $i_{v_{\beta_i}}$)

$$(\blacklozenge) = i_{v_{\beta_k}} - i_{v_{\beta_1}}\mathcal{L}_v\omega = 0 \qquad \square$$

* Conserved quantities from homotopy comomentum maps

(M, ω) pre n-plectic

G acting on M $\quad$ $\mathcal{V}$: right action $\quad$ $\mathcal{V}$ multisymplectic

$\mathfrak{g}$ = Lie G $\quad$ $\mathcal{V}_g^* \omega = \omega \quad \mathcal{V}_g = \mathcal{V}(\cdot, g)$

infinitesimally we have a Lie algebra homomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{X}(M) \qquad x \longmapsto v_x \qquad \mathcal{L}_{v_x} \omega = 0 \quad \forall x \in \mathfrak{g}$$

(multisymplecticity)

For G connected $\mathcal{V}$ is multisymplectic $\Leftrightarrow$ its infinitesimal action is multisymplectic

$$v_x(m) = \frac{d}{dt}\Big|_{t=0} \mathcal{V}(m, \exp(tx)) \qquad \forall m \in M$$

* multisymplectic infinitesimal action: homomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{X}(M, \omega) = \{ X \in \mathfrak{X}(M) \,/\, \mathcal{L}_X \omega = 0 \}$$

$\rightsquigarrow$ get an "L_∞-lift" to $L_\infty(M, \omega)$

$\mathfrak{g}$: Lie algebra $\quad$ Homology differential ∂

$$\partial_k : \partial\Big|_{\Lambda^k \mathfrak{g}} : \Lambda^k \mathfrak{g} \longrightarrow \Lambda^{k-1} \mathfrak{g} \qquad k \geq 1$$

$$x_1 \wedge x_2 \wedge \dots \wedge x_k \xrightarrow{\ \partial\ } \sum_{1 \leq i < j \leq k} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \dots \wedge \underset{\text{deletion}}{\hat{x}_i} \wedge \dots \wedge x_k$$

$$\partial_0 : \underset{\mathbb{R}}{\underset{\parallel}{\Lambda^0 \mathfrak{g}}} \longrightarrow \Lambda^{-1} \mathfrak{g} = \{0\} \qquad \partial_0 = 0\text{-map}$$

★ L_∞ - morphism (f) from $\mathfrak{g}$ to $L_\infty(M,\omega)$

$$\boxed{(f) = \{ f_i : \Lambda^i \mathfrak{g} \to \Omega^{n-i}(M) \;/\; 1 \le i \le n \}} \quad \text{with}$$

$$(\blacklozenge) \quad \boxed{\begin{array}{l} \operatorname{Im} f_1 \subset \Omega^{n-1}_{Ham}(M) \\[1em] -f_{k-1}(\partial(p)) = df_k(p) + \varsigma(k)\, i_{v_p}\omega \end{array}} \quad (*)$$

$\varsigma(k) = -(-1)^{k(k+1)/2}$

$k = 1 \dots n+1$, $p \in \Lambda^k \mathfrak{g}$ $\quad (f_0 = f_{n+1} = 0)$

$v_p : v_{x_1} \wedge \dots v_{x_k}$ if $p = x_1 \wedge x_2 \wedge \dots \wedge x_k$ $\quad x_i \in \mathfrak{g}$

(*) analogy
homotopy operators in homology & cohomology
$d h + h d = id$

★ **Homotopy co-momentum map**
for $v: \mathfrak{g} \to \mathcal{X}(M)$ on (M,ω):

L_∞ - morphism $(f) : \mathfrak{g} \to L_\infty(M,\omega)$

s.t. $\forall\, x \in \mathfrak{g}$,

$$(\blacklozenge\blacklozenge) \quad \boxed{d(f_1(x)) = -i_{v_x}\omega}$$

$k=1$
$\varsigma(k) = -1$
$f_{k-1} = f_0 = 0$

(... Chern complex picture emerging)

(f) is G-equivariant if $f_i : \Lambda^i \mathfrak{g} \to \Omega^{n-i}(M)$ are such

For G connected, we have the following **equivalent** infinitesimal formulation

$$\forall q \in \Lambda^i \mathfrak{g}, \; \forall x \in \mathfrak{g} = T_e G, \quad \boxed{\mathcal{L}_{v_x}[f_i(q)] = f_i([x,q])}$$

$[x, \bullet] = ad(x)$ on $\Lambda^\bullet \mathfrak{g}$

Let $H \in \Omega^{n-1}(M)$ (Hamiltonian n-form)

Let $\mathfrak{g} \to \mathcal{X}(M,\omega)$, $x \mapsto v_x$ be an infinitesimal action

This action is called

(a) locally H-preserving if $\mathcal{L}_{v_x} H$ is closed $\forall x \in \mathfrak{g}$

(b) globally H-preserving if $\mathcal{L}_{v_x} H$ is exact $\forall x \in \mathfrak{g}$

(c) strictly H-preserving if $\mathcal{L}_{v_x} H = 0$ $\forall x \in \mathfrak{g}$

"H-preserved by an infinitesimal action"

$\boxed{\text{Case (a)}}$ The generator of the infinitesimal action is $f_1(x)$ (by definition)

Let v_H be an Hamiltonian v.f. for H

We have the following **Lemma**

(i) $f_1(x) \in C_{loc}(v_H)$ $\forall x \in \mathfrak{g}$

(ii) $i_{[v_H, v_x]}\omega = 0$ $\forall x \in \mathfrak{g}$

(iii) $i(v_p)\omega \in C_{str}(v_H)$ $\forall p \in \Lambda^k \mathfrak{g}$

Ad (i) Recall that $\alpha \in \Omega^{n-1}(M)$ is l.c. by $v_H \Leftrightarrow \mathcal{L}_{v_\alpha} H$ is closed $\forall v_\alpha$, ham. v.f. for α, and use $df_1(x) = -i_{v_x}\omega$ — or use (+) below (involving (ii))

see RW2

Ad (ii) $\alpha \in \Omega^{n-1}(M)$ is g.c. iff $i_{[v_H, v_\alpha]}\omega = 0$

Ad (iii) Recall $[\mathcal{L}_v, i_w] = i_{[v,w]}$. Then compute

$$\boxed{\mathcal{L}_{v_H}[i(v_p)\omega]} = \mathcal{L}_{v_H}(i_{v_{x_k}} \cdots i_{v_{x_1}}\omega) = -i_{v_{x_k}}\mathcal{L}_{v_H} \cdots i_{x_1}\omega = \pm i(v_p)\mathcal{L}_{v_H}\omega$$

$$\boxed{0}$$

$$(+)\quad d\,\mathcal{L}_{v_H} f_1(\alpha) = \mathcal{L}_{v_H} df_1(\alpha) = -\mathcal{L}_{v_H} i_{v_\alpha}\omega = -i_{v_\alpha}\mathcal{L}_{v_H}\omega = 0$$

(since $i_{[v_\alpha, v_H]}\omega = 0$)

★ Lie algebra homology

$\mathfrak{g}$ Lie algebra, $k \geq 1$ $\quad \partial_k$: Lie homology differential

(a) $Z_k(\mathfrak{g}) = \operatorname{Ker}(\partial_k) \subset \Lambda^k \mathfrak{g}$ cycles

(b) $B_k(\mathfrak{g}) = \operatorname{Im}(\partial_{k+1}) \subset \Lambda^k \mathfrak{g}$ boundaries

(c) $H_k(\mathfrak{g}) = Z_k(\mathfrak{g}) / B_k(\mathfrak{g})$ homology space

$Z_k(\mathfrak{g}) \equiv P_{\mathfrak{g},k} \equiv$ k-th Lie kernel

• conserved quantities from globally & strictly H-preserving actions
• Homological viewpoint
• construction of preserved Hamiltonians

Proposition

Let $p \in Z_k(\mathfrak{g})$. Then $f_k(p)$ is locally conserved by any v_H

$k=1$ has been already proven. Let $k>1$.

Compute: use (★), page XVIII-9 Here $\partial p = 0$

$$\boxed{d \underbrace{\mathcal{L}_{v_H} f_k(p)} = \mathcal{L}_{v_H} d f_k(p) = -\mathcal{L}_{v_H} i_{v_x} \omega = \pm i_{v_x} \mathcal{L}_{v_H} \omega = 0}$$

recall $i_{[v_H, v_x]} \omega = 0$

$\Rightarrow$ we get a closed $(n-k)$-form

If $H^{n-k}_{dR}(M) = 0$, $\mathcal{L}_{v_H} f_k(p)$ is exact, i.e. $f_k(p)$ is globally conserved

★ **Theorem** $\quad p \in B_k(\mathfrak{g}) \Rightarrow f_k(p)$ globally conserved

⚠ $\partial p = 0$ is not enough to ensure this

Indeed, let $p = \partial_{k+1} q$. Then

$$\boxed{\mathcal{L}_{v_H}(f_k(p))} = \mathcal{L}_{v_H}(f_k(\partial q)) = \mathcal{L}_{v_H}(-d f_{k+1}(q) - \varsigma(k+1) i(v_q)\omega)$$

$$= -d \mathcal{L}_{v_H} f_{k+1}(q) - \varsigma(k+1) \underbrace{\mathcal{L}_{v_H} i(v_q) \omega}_{\substack{\shortparallel \\ 0}} = \boxed{-d \mathcal{L}_{v_H} f_{k+1}(q)}$$

yielding the conclusion.

⚠ $\boxed{\text{Notice that}}$ $\partial p = 0 \not\Rightarrow f_k(p)$ $\boxed{\text{globally conserved}}$

Indeed: $M = \mathbb{R}^3$ $\quad \omega = dx \wedge dy \wedge dz$

$H = -x\,dy$

Then (already done) $dH = -dx \wedge dy$,

$v_H = \frac{\partial}{\partial z}$

Let $\mathfrak{g} = \langle a, b \rangle_{\mathbb{R}}$ & $v : \mathfrak{g} \to \mathfrak{X}(M)$

$$\boxed{v_a = \frac{\partial}{\partial x} \qquad v_b = \frac{\partial}{\partial y}}$$

$$\boxed{\mathcal{L}_{v_a} H = i_{v_a} dH + d\, i_{v_a} H = i_{\frac{\partial}{\partial x}}(-dx \wedge dy) + d \cdot 0 = -dy}$$

(exact)

$$\boxed{\mathcal{L}_{v_b} H = i_{\frac{\partial}{\partial y}}(-dx \wedge dy) + d(-x) = dx - dx = 0}$$

- comomentum map! $\quad f_1(a) = -y\,dz$

$f_1(b) = x\,dz$

$f_2(a \wedge b) = -z$

Let us check the conditions required by the definition

$$df_1(a) = -dy \wedge dz = -i_{\frac{\partial}{\partial x} = a}(dx \wedge dy \wedge dz)$$ This is $(\ast\ast)$

$$df_1(b) = dx \wedge dz = -i_{\frac{\partial}{\partial y} = b}(dx \wedge dy \wedge dz)$$

clearly $\operatorname{Im} f_k \subset \Omega^1_{Ham}(\mathbb{R}^3)$

$a \wedge b \in Z_2(\mathfrak{g})$, $-z$ is locally conserved (w.r.to v_H)

$$d\, \mathcal{L}_{\frac{\partial}{\partial z}}(-z) = d(-1) = 0$$

☆ $\mathcal{L}_{\frac{\partial}{\partial z}}(-z) = -1$ is not exact

Check that

$$\boxed{\underbrace{-f_1(\partial p)}_{\overset{\text{"}}{0}} \overset{?}{=} df_2(p) + \varsigma(2)\, i_{a \wedge b}(dx \wedge dy \wedge dz)} \quad k=2$$

$$= -dz + [\underbrace{-(-1)^{\frac{2\cdot 3}{2}}}_{+1}] \cdot \underbrace{i_{\frac{\partial}{\partial y}}(\underbrace{i_{\frac{\partial}{\partial x}} dx \wedge dy \wedge dz}_{dy \wedge dz})}_{dz} = -dz + dz = 0$$

☆ Conserved quantities from globally H-preserving actions

$\boxed{\text{Case (b)}}$

(M,ω) pre n-plectic manifold

$H \in \Omega^{n-1}_{Ham}(M)$

$(f) : \mathfrak{g} \to L_\infty(M,\omega)$ comomentum of a globally H-preserving infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M,\omega)$

$x \mapsto v_x$

! In view of the preceding example, no significant improvements are expected; specifically

$f_1(x) \in C(v_H) \quad \forall x \in \mathfrak{g}$ and for any v_H

$(\mathcal{L}_{v_H} f_1(x)$ exact$)$ [Same proof of the prec. lemma]

☆ Conserved quantities from strictly H-preserving actions

$\boxed{\text{Case (c)}}$

(f) comomentum of a strictly H-preserving inf. action $\mathfrak{g} \to \mathfrak{X}(M,\omega)$

$x \mapsto v_x$

this can be realised if a compact Lie group G acting on M

Lemma: For any form Ω on M, $\forall m \geq 1$ & $v_1 \ldots v_m \in \mathfrak{X}(M)$

(proof omitted: see Madsen-Swann)

$$\boxed{(-1)^m d\, i(v_1 \wedge \_ v_m)\Omega = i(\partial(v_1 \wedge \_ v_m))\Omega + \sum_{i=1}^m (-1)^i\, i(v_1 \wedge \_ \hat{v}_i \wedge . v_m)\mathcal{L}_{v_i}\Omega + i(v_1 \wedge \_ v_m)\, d\Omega}$$

upon defining

$$\boxed{\mathcal{L}_Y \Omega := d i_Y \Omega - (-1)^m i_Y d\Omega}$$

Y = multiv. field,

we may rephrase the formula in the following guise

$$\boxed{\mathcal{L}_{v_1 \wedge .. v_m} \Omega = (-1)^m i(\partial(v_1 \wedge \_ v_m))\Omega + \sum_{i=1}^m (-1)^i\, i(v_1 \wedge .. \hat{v}_i \wedge .. v_m)\mathcal{L}_{v_i}\Omega}$$

☆ Generalized Cartan formula

Thm. Let $p \in Z_k(\mathfrak{g})$. Then $f_k(p)$ is globally conserved

Proof. $\boxed{i_{v_H} df_k(p)} = -\varsigma(k)\, i_{v_H}\, i(v_p)\omega$

$= (-1)^k \varsigma(k)\, i_{v_p}\, dH$

$= \boxed{\varsigma(k)\, d\,(i(v_p)H)}$

recall:
$-f_{k-1}(\partial p) = df_k(p) + \varsigma(k)\, i(v_p)\omega$

The preceding lemma, $\partial p = 0$, strict invariance of H

$\Rightarrow$

$$\mathcal{L}_{v_H} f_k(p) = d\left[\, i_{v_H} f_k(p) + \varsigma(k)\, i(v_p) H \,\right]$$

Remark: in the symplectic case we recover standard (co)moment maps: $f_1 : \mathfrak{g} \to \Omega^0(M)$

In view of $Z_1(\mathfrak{g}) = \mathfrak{g}$, we recover "if H is $\mathfrak{g}$-invariant, then $\forall x \in \mathfrak{g}$ we have $\{f_1(x), H\} = \mathcal{L}_{v_H} f_1(x) = 0$"

⚠ $f_1(x)$ is NOT strictly conserved, in general (even if $x \in B_1(\mathfrak{g})$):

$M = \mathbb{R}^3 \quad \omega = dx \wedge dy \wedge dz \quad H = -x\,dy \quad v_H = \frac{\partial}{\partial z}$

$\alpha = z\,dx \qquad \mathbb{R}\text{-action: } 1 \mapsto v_\alpha = -\frac{\partial}{\partial y}$

★ comomentum map! $f_1(1) = \alpha$

Then $\boxed{\mathcal{L}_{v_\alpha} H} = i_{v_\alpha} dH + d\, i_{v_\alpha} H = i_{-\frac{\partial}{\partial y}}(-dx \wedge dy) + d\left[i_{-\frac{\partial}{\partial y}}(-x\,dy)\right]$

$= -dx + dx \;\boxed{= 0}$

but $\boxed{\mathcal{L}_{v_H}\alpha} = \mathcal{L}_{\frac{\partial}{\partial z}}(z\,dx) = i_{\frac{\partial}{\partial z}}(dz \wedge dx) + d\left[i_{\frac{\partial}{\partial z}} z\,dx\right]$

$= +dx + 0 = \boxed{dx \neq 0}$

☆ Homological viewpoint

Setting: $\mathfrak{g}$ acting on pre n-plectic (M,ω)

$H \in \Omega^{n-1}(M)$, v_H H. v. field

locally H-preserving: $d[\mathcal{L}_{v_x} H] = 0$

The map

$$(☆) \qquad \boxed{\mathfrak{g} \ni x \longmapsto [\mathcal{L}_{v_x} H] \in H^{n-1}_{dR}(M)}$$

measures obstruction to a global H-preserving action

This map is zero on $[\mathfrak{g},\mathfrak{g}]^{(+)}$, so it descends to $H_1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$

(+) see (++) below

and can be extended to the whole Lie algebra homology

★ **Theorem**: for $k = 1, 2, \dots \dim(\mathfrak{g})$, the map

$$A: H_k(\mathfrak{g}) \longrightarrow H^{n-k}_{dR}(M)$$
$$[p] \longmapsto [\mathcal{L}_{v_p} H]$$

is well-defined

Let $p \in Z_k(\mathfrak{g})$. Check that $d(\mathcal{L}_{v_p} H) = 0$. Set $v_p = \sum_\ell v_1^\ell \wedge \dots v_k^\ell$,

we get

$$\boxed{d\mathcal{L}_{v_p} H} = (-1)^{k+1} \mathcal{L}_{v_p} dH$$

Lie derivative for Y m-multiv. f.

$$\mathcal{L}_Y \Omega := d\, i_Y \Omega - (-1)^m i_Y d\Omega$$

One has

$$\mathcal{L}_{v_1\wedge\dots v_m}\Omega = (-1)^m i(\partial(v_1\wedge\dots v_m)\Omega) + \sum_{i=1}^m (-1)^i i(v_1\wedge\dots \hat{v_i}\wedge\dots v_m)\mathcal{L}_{v_i}\Omega$$

$$= -(i(v_{\partial p}) dH + \sum_\ell \sum_{i=1}^k (-1)^i i(v_1^\ell \wedge \dots \hat{v_i^\ell} \dots v_m^\ell) \mathcal{L}_{v_i^\ell} dH$$

swap

$$= \boxed{0}$$

$(\partial p = 0$, $d\mathcal{L}_{v_i} H = 0)$

Subsequently, let $q \in \Lambda^{k+1}\mathfrak{g}$, $v_p := \sum_\ell v_1^\ell \wedge \dots v_{k+1}^\ell$

(++) Check that $\mathcal{L}_{v_{\partial q}} H$ is exact: $\mathcal{L}_{v_{\partial q}} H = \underbrace{d(i_{v_{\partial q}} H)}_{\text{already exact}} \pm i_{v_{\partial q}} dH$

let us check that this is exact

Then notice that

$$d\mathcal{L}_{v_q} H = \mathcal{L}_{v_q} dH = \pm i(\underbrace{\partial v_q}_{v_{\partial q}}) dH + \sum (-1)^i (v_1 \dots \hat{v_i} \dots) \underbrace{\mathcal{L}_{v_i} dH}_{d\mathcal{L}_{v_i} H = 0}$$

$= \pm i(v_{\partial q}) dH$

and we are done □

If (f) is a comomentum map, A admits the following explicit representation $\forall p \in Z_k(\mathfrak{g})$ $(\partial p = 0)$

$$\boxed{A[p] = -\varsigma(k)\left[\mathcal{L}_{v_H} f_k(p)\right]}$$

Indeed, $A[p] = [\mathcal{L}_{v_p} H] = (-1)^r [\, i(v_p)\, i_{v_H}\omega\,]$

$= i_{v_H} i_{v_p}\omega = -\varsigma(k)\, i_{v_H} df_k(p) \overset{\text{def. of comomentum map}}{=}$

$= -\varsigma(k)\big(\underbrace{-d\, i_{v_H} f_k(p)}_{\text{ } \mapsto 0 \text{ in cohomology}} + \mathcal{L}_{v_H} f_k(p)\big)$

$\Rightarrow$ the conclusion follows, upon passage to cohomology.

$\star$ Comments: $f_k(p)$ l.c. when $p \in Z_k(\mathfrak{g})$
g.c when $p \in B_k(\mathfrak{g})$

as it was already known

The map J : $C_{loc}(v_H)/C(v_H) \hookrightarrow H_{dR}(M)$

$[\alpha] \longmapsto [\mathcal{L}_{v_H}\alpha]$

is canonical & injective.

We have the factorization

$$\boxed{\begin{tikzcd} H_k(\mathfrak{g}) \arrow[r, "-\varsigma(k)\cdot"] \arrow[rr, bend right, "A"'] & C^{n-k}_{loc}(v_H)/C^{n-k}(v_H) \arrow[r, "J"] & H^{n-k}_{dR}(M) \end{tikzcd}}$$

If the action is strictly H-preserving, $f_k(p)$ is globally conserved $\forall p \in Z_k(\mathfrak{g})$ (known)

Further remarks

(1) If the action is **globally** H-preserving,

$\mathfrak{g} \to H^{n-1}_{dR}(M)$ is the zero map.

However, the higher components do not vanish in general.

Recall the previous example

$i(v_a \wedge v_b)\, dH = -i(\partial_x \wedge \partial_y)(dx \wedge dy) = -1$

is closed but not exact

(2) If the action is **strictly** H-preserving, then $\mathbf{A = 0}$

☆ Preserved Hamiltonians

$\vartheta : M \times G \to M$ smooth action of G (Lie, connected) compact
on (M, ω) pre n-plectic, $\vartheta_g^* \omega = \omega \quad \forall g \in G$

☆ Theorem

Let $\hat{\tilde{H}} \in \Omega^{n-1}_{Ham}(M)$ locally preserved by the action $\boxed{d \mathcal{L}_\xi \tilde{H} = 0}$

Then $\exists$ $H \in \Omega^{n-1}_{Ham}(M)$ strictly preserved and

such that any $v_{\tilde{H}}$ is a v_H (Hamiltonian v.f.)

Proof ("Weyl trick"). Set $H = \int_G \vartheta_g^*(\tilde{H}) \cdot dg$ normalized Haar measure

H is strictly preserved. Then we have

$$\boxed{dH} = \int_G \theta_g^*(d\tilde{H}) \cdot dg = \int_G d\tilde{H} \cdot dg = \boxed{d\tilde{H}}$$

$0 = d\mathcal{L}_\xi \hat{\tilde{H}} = \mathcal{L}_\xi d\tilde{H}$

$\Rightarrow \theta_g^*(d\tilde{H}) = d\tilde{H}$

and this yields the conclusion □

☆ Thm

Let v G-invariant on M, $\mathcal{L}_v \omega = 0$. Let $H^n_{dR}(M) = 0$

Then $v = v_H$ for a G-invariant H (strictly preserved)

Proof $\mathcal{L}_v \omega = 0 \Rightarrow i_v \omega$ closed $\Rightarrow i_v \omega$ exact

$i_v \omega = -d\tilde{H}$ for some $\tilde{H} \in \Omega^{n-1}_{Ham}(M)$. The form

$i_v \omega$ is G-invariant $\Rightarrow H = \int_G \vartheta_g^* H$ satisfies

$i_v \omega = -dH$.

In particular, let ω be a volume form on M. If v is G-invariant and divergence-free ($\mathcal{L}_v \omega = 0$). If $H^{\dim(M)-1}_{dR}(M) = 0$, then $v = v_H$, H G-invariant Hamiltonian form

(Example: M compact + simply connected (use Poincaré duality...) is ok

Thm: Let G be connected, acting on (M, ω) pre n-plectic.
Let $v \in \mathfrak{X}(M)$ be G-invariant. Let $H^n_{dR}(M) = 0$
$= H^{n-1}_{dR}(M)$

Then $v = v_H$, for H globally preserved.

Proof $\quad \mathcal{L}_v \omega = 0 \Rightarrow d\, i_v \omega = 0 \overset{H^n = 0}{\Rightarrow} i_v \omega = -dH$

(H non nec. G-invariant). $i_v \omega$ is G-invariant, therefore

$0 = \mathcal{L}_{\underset{\in \mathfrak{g}}{v_x}} dH = d\, \mathcal{L}_{v_x} H$. Hence ($H^{n-1} = 0$)
$\mathcal{L}_{v_x} H$ is exact
i.e. H is globally preserved.

TOPICS IN SYMPLECTIC AND MULTISYMPLECTIC GEOMETRY Ph.D. Course

Prof. M. Spera UCSC, Brescia

Lecture XX

- An introduction to covariant phase space (à la Forger-Romero)
- particle mechanics calculation

★★ Symplectic structure on covariant phase space

(Crnković-Witten, Zuckerman, Forger-Romero, Marsden et al)

★ Outline

multisymplectic approach (à la F-R) via $\omega_{\mathcal{L}}$ & $\omega_{\mathcal{H}}$

- Lagrangian approach $F = E$ (conf. bundle)
- Hamiltonian approach $F = \vec{J}^{\circledast} E$ (multiphase space)

$\phi \in \mathcal{C}$ (a section of $F \to M$)

various options for $T_\phi \mathcal{C}$ (tangent space)

$$\left[\, S = \{ \phi \in \mathcal{C} \,/\, \mathcal{S}'[\phi] = 0 \} \,\right] \qquad \mathcal{S} : \text{action}$$

★★ covariant phase space

$T_\phi S$ solutions of the linearized equation of motion (Jacobi equation)

$$\boxed{T_\phi S = \ker \mathcal{J}[\phi]}$$

$$\boxed{\mathcal{J}[\phi] : \Gamma(\phi^* VF) \longrightarrow \Gamma(\phi^* V^{\circledast} F)} \quad ★ \text{ Jacobi operator}$$

★ **Symplectic structure** • Lagrangian picture

$$\Theta_\phi(\delta\phi) = \int_\Sigma (\varphi, \partial\varphi)^* \, \underbrace{\Theta_{\mathcal{L}}(\delta\varphi, \partial\delta\varphi)}$$

Σ : "Cauchy surface"

$$\Omega_\phi(\delta\phi_1, \delta\phi_2) = \int_\Sigma (\varphi, \partial\varphi)^* \, \omega_{\mathcal{L}}(\delta\varphi_1, \partial\delta\varphi_1; \delta\varphi_2, \partial\delta\varphi_2)$$

Operationally: insert $\delta\phi$ in the first slot of $\Theta_{\mathcal{L}}$ (an n-form) then pull-back ($\partial\phi$ is involved..), get an $(n-1)$-form which is to be integrated along Σ, $(n-1)$-submanifold of M

similarly for Ω_ϕ.

Explicitly:

$$\boxed{\Theta_\phi(\delta\phi) = \int_\Sigma d\sigma_\mu \frac{\partial L}{\partial q^i_\mu}(\varphi, \partial\varphi)\, \delta\varphi^i}$$

(see F-R)

⚠ J is different in Crnković-Witten...

$$\boxed{\Omega_\phi(\delta\phi_1, \delta\phi_2) = \int_\Sigma d\sigma_\mu\, J^\mu_\phi(\delta\phi_1, \delta\phi_2)}$$

"current"

$$\boxed{\begin{aligned} J^\mu_\phi(\delta\phi_1, \delta\phi_2) &= \frac{\partial^2 L}{\partial q^j \partial q^i_\mu}(\varphi, \partial\varphi)\left(\delta\varphi^i_1 \delta\varphi^j_2 - \delta\varphi^i_2 \delta\varphi^j_1\right) \\ &+ \frac{\partial^2 L}{\partial q^j_\nu \partial q^i_\mu}(\varphi, \partial\varphi)\left(\delta\varphi^i_1 \partial_\nu \delta\varphi^j_2 - \delta\varphi^i_2 \partial_\nu \delta\varphi^j_1\right)\end{aligned}}$$

★ Symplectic structure • Hamiltonian picture

$$\boxed{\Theta_\phi(\delta\phi) = \int_\Sigma (\varphi, \pi)^* \Theta_{\mathcal{H}}(\delta\varphi, \delta\pi)}$$

$$\boxed{\Omega_\phi(\delta\phi_1, \delta\phi_2) = \int_\Sigma (\varphi, \pi)^* \Omega_{\mathcal{H}}(\delta\varphi, \delta\pi) = \int_\Sigma d\sigma_\mu J^\mu_\phi(\delta\phi_1, \delta\phi_2)}$$

$$\boxed{J^\mu_\phi(\delta\phi_1, \delta\phi_2) = \delta\varphi^i_1\, \delta\pi_{2,i}{}^\mu - \delta\varphi^i_2\, \delta\pi_{1,i}{}^\mu}$$

"current" (independent of $\mathcal{H}$, as it should be)

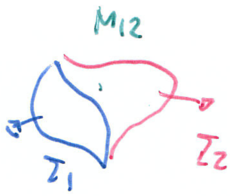
In both cases $\boxed{\partial_\mu J^\mu = 0}$ current conservation

and this shows that Ω is independent of the choice of Σ

$$\boxed{\Omega_{\Sigma_2} = \Omega_{\Sigma_1} \quad ; \quad \Theta_{\Sigma_2} - \Theta_{\Sigma_1} = \delta S_{M_{1,2}}}$$

exterior differential...

see also Pélan 2011



☆ Details on Jacobi operators

☆ Lagrangian picture

$$\varphi = \varphi_\lambda\big|_{\lambda=0} \qquad \delta\varphi = \frac{\partial \varphi_\lambda}{\partial \lambda}\Big|_{\lambda=0}$$

Solution of E-L

Evaluate $\mathcal{D}_\mathcal{L}(\varphi_\lambda, \partial\varphi_\lambda, \partial^2\varphi_\lambda)$ (section of $\varphi_\lambda^* V^{\circledast} E$

recall:

$$\mathcal{D}_\mathcal{L}(\varphi_\lambda) = \left(\partial_\mu \left(\frac{\partial L}{\partial q^i_\mu}(\varphi_\lambda, \partial\varphi_\lambda) \right) - \frac{\partial L}{\partial q^i}(\varphi_\lambda, \partial\varphi_\lambda) \right) dq^i \otimes d^n x$$

Then

$$\frac{\partial \mathcal{D}_\mathcal{L}(\varphi_\lambda)}{\partial \lambda}\Big|_{\lambda=0} = \underbrace{\overset{\text{vertical}}{\delta\varphi^i \frac{\partial}{\partial q^i}}}_{\substack{\text{can be removed}\\ \text{to get something on } M}} + \underbrace{\left\{ \partial_\mu \left[\frac{\partial^2 L}{\partial q^j \partial q^i_\mu} \delta\varphi^j + \frac{\partial^2 L}{\partial q^j_\nu \partial q^i_\mu} \partial_\nu \delta\varphi^j \right] - \frac{\partial^2 L}{\partial q^j \partial q^i} \delta\varphi^j - \frac{\partial^2 L}{\partial q^j_\nu \partial q^i} \partial_\nu \delta\varphi^j \right\} dq^i \otimes d^n x}_{\text{This will be } \mathcal{J}[\phi]}$$

see F-R for an intrinsic procedure

to be equated to 0 (Jacobi equation)

☆ Hamiltonian picture

$$\frac{\partial}{\partial \lambda} \mathcal{D}_\mathcal{H}(\varphi_\lambda, \pi_\lambda, \partial\varphi_\lambda, \partial\pi_\lambda)\Big|_{\lambda=0} =$$

$$\underbrace{\delta\varphi^i \frac{\partial}{\partial q^i} + \delta\pi^\mu_i \frac{\partial}{\partial p_i^\mu}}_{\substack{\text{vertical}\\ \text{it can be removed}}} + \underbrace{\left\{ \frac{\partial^2 H}{\partial q^j \partial q^i} \delta\varphi^j + \frac{\partial^2 H}{\partial p^\nu_j \partial q^i} \delta\pi^\nu_j + \partial_\mu \delta\pi^\mu_i \right\} dq^i \otimes d^n x + \left\{ \frac{\partial^2 H}{\partial q^j \partial p^\mu_i} \delta\varphi^j + \frac{\partial^2 H}{\partial p^\nu_j \partial p^\mu_i} \delta\pi^\nu_j - \partial_\mu \delta\varphi^i \right\} dp^\mu_i \otimes d^n x}_{\text{this will be } \mathcal{J}[\phi]}$$

again see FR

to be equated to 0 (J - equation)

★ (Towards) covariant phase space (I)

Jacobi equation & Noether current conservation in particle mechanics (Lagrangian approach)

Simplest example

☆ 1 degree of freedom

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0 \qquad E-L \qquad \frac{d}{dt}\delta q = \delta \dot q$$

~ get Jacobi equation from $q = q_\lambda$, $\delta q = \left.\frac{d q_\lambda}{d\lambda}\right|_{\lambda=0}$

$\delta \dot q = \left.\frac{d \dot q_\lambda}{d\lambda}\right|_{\lambda=0}$

via differentiation at $\lambda = 0$

$$0 = \frac{d}{dt}\left[\frac{\partial^2 L}{\partial q \partial \dot q}\delta q + \frac{\partial^2 L}{\partial \dot q \partial \dot q}\delta \dot q\right]$$

$$- \frac{\partial^2 L}{\partial q \partial q}\delta q - \frac{\partial^2 L}{\partial \dot q \partial q}\delta \dot q = \frac{d}{dt}\left(\frac{\partial^2 L}{\partial q \partial \dot q}\right)\delta q + \frac{\partial^2 L}{\partial q \partial \dot q}\delta \dot q$$

cancel out

$$+ \frac{d}{dt}\left(\frac{\partial^2 L}{\partial q \partial \dot q}\right)\delta \dot q + \frac{\partial^2 L}{\partial q \partial \dot q}\delta \ddot q - \frac{\partial^2 L}{\partial q \partial q}\delta q - \frac{\partial^2 L}{\partial \dot q \partial q}\delta \dot q$$

Regrouping terms:

★ Jacobi equation

$$(\star) \quad \boxed{\frac{\partial^2 L}{\partial \dot q \partial \dot q}\delta \ddot q = \left[\frac{\partial^2 L}{\partial q \partial q} - \frac{d}{dt}\frac{\partial^2 L}{\partial q \partial \dot q}\right]\delta q - \frac{d}{dt}\left[\frac{\partial^2 L}{\partial \dot q \partial \dot q}\right]\delta \dot q}$$

non singular...

(simplified...)

[Remark: working out the full calculation for $L = \frac{1}{2} g_{ij}\dot q^i \dot q^j$ one gets Jacobi equation in Riemannian geometry]

Symplectic current (Forger - Romero)

$$J(\delta q_1, \delta q_2) = \frac{\partial^2 L}{\partial q \partial \dot q}\underbrace{(\delta q_1 \delta q_2 - \delta q_2 \delta q_1)}_{=0} + \frac{\partial^2 L}{\partial \dot q \partial \dot q}(\delta q_1 \delta \dot q_2 - \delta \dot q_1 \delta q_2)$$

solutions of Jacobi

Compute

$$\frac{dJ}{dt} = \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot q \partial \dot q}\right)(\delta q_1 \delta \dot q_2 - \delta q_2 \delta \dot q_1) + \frac{\partial^2 L}{\partial \dot q \partial \dot q}\left[\delta \dot q_1 \delta \dot q_2 + \delta q_1 \delta \ddot q_2 - \delta \dot q_2 \delta \dot q_1 - \delta q_2 \delta \ddot q_1\right]$$

cancel out

Then use $(\star)$. We find

$$\boxed{\frac{dJ}{dt}} = \frac{d}{dt}\left(\frac{\partial^2 L}{\partial\dot q\,\partial\dot q}\right)\left[\delta q_1\,\delta\dot q_2 - \delta q_2\,\delta\dot q_1\right]$$

$$+\ \delta q_1\left[\left(\frac{\partial^2 L}{\partial q\,\partial q} - \frac{d}{dt}\frac{\partial^2 L}{\partial q\,\partial\dot q}\right)\delta q_2 - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial\dot q\,\partial\dot q}\right)\delta\dot q_2\right]$$

$$-\ \delta q_2\left[\frac{\partial^2 L}{\partial q\,\partial q} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial q\,\partial\dot q}\right)\delta q_1 - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial\dot q\,\partial\dot q}\right)\delta\dot q_1\right]$$

Check cancellations...

$$\boxed{= 0}$$

★ (Towards) covariant phase space (II)

Jacobi equation & Noether current conservation in particle mechanics (Hamiltonian approach) in $\mathbb{R}^2$

Start from Hamilton's equations & variate: $\left[q = q_\lambda \quad \delta q = \frac{d}{d\lambda} \delta q_\lambda \Big|_{\lambda=0} \text{ et cetera} \right]$

$$\boxed{\begin{cases} \dot q = \dfrac{\partial H}{\partial p}(q,p) \\ \dot p = -\dfrac{\partial H}{\partial q}(q,p) \end{cases}} \rightsquigarrow \begin{cases} \dot{\overbrace{q+\delta q}} = \dfrac{\partial H}{\partial p}(q+\delta q, p+\delta p) \\ \dot{\overbrace{p+\delta p}} = -\dfrac{\partial H}{\partial q}(q+\delta q, p+\delta p) \end{cases}$$

[equiv. differentiate for q_λ & p_λ and evaluate at $\lambda = 0$]

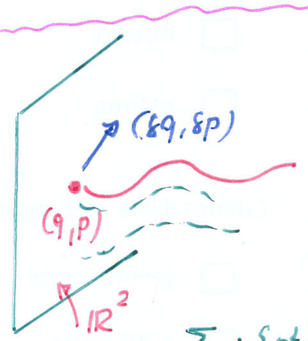
Hamilton eq. enforced

expand at first order in $\delta q, \delta p$

$$\begin{cases} \dot q + \dot{\widehat{\delta q}} = \dfrac{\partial H}{\partial p}(q,p) + \dfrac{\partial^2 H}{\partial q \partial p}(q,p)\,\delta q + \dfrac{\partial^2 H}{\partial p \partial p}(q,p)\,\delta p + \dots \\ \dot p + \dot{\widehat{\delta p}} = -\dfrac{\partial H}{\partial q}(q,p) - \dfrac{\partial^2 H}{\partial q \partial q}(q,p)\,\delta q - \dfrac{\partial^2 H}{\partial p \partial q}(q,p)\,\delta p + \dots \end{cases}$$

★★ Jacobi equation

$$\boxed{\begin{pmatrix} \dfrac{d}{dt}\delta q \\ \dfrac{d}{dt}\delta p \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 H}{\partial q \partial p} & \dfrac{\partial^2 H}{\partial p \partial p} \\ -\dfrac{\partial^2 H}{\partial q \partial q} & \dfrac{\partial^2 H}{\partial p \partial q} \end{pmatrix} \begin{pmatrix} \delta q \\ \delta p \end{pmatrix}}$$



$\Sigma : \{pt\}$

"Cauchy surface" ~ initial conditions

in the present case no integration occurs

Compute

$$\boxed{\frac{d}{dt}} \underbrace{(\delta q_1 \delta p_2 - \delta q_2 \delta p_1)}_{J} = \frac{d}{dt}(\delta q_1)\,\delta p_2 + \delta q_1 \frac{d}{dt}\delta p_2$$

$$- \frac{d}{dt}(\delta q_2)\,\delta p_1 - \delta q_2 \frac{d}{dt}\delta p_1 =$$

J (symplectic form on $\mathbb{R}^{2n}$)

$$= \left(\frac{\partial^2 H}{\partial q \partial p}\delta q_1 + \frac{\partial^2 H}{\partial p \partial p}\delta p_1 \right)\delta p_2 + \delta q_1 \left(-\frac{\partial^2 H}{\partial q \partial q}\delta q_2 - \frac{\partial^2 H}{\partial p \partial q}\delta p_2 \right)$$

$$- \left(\frac{\partial^2 H}{\partial q \partial p}\delta q_2 + \frac{\partial^2 H}{\partial p \partial p}\delta p_2 \right)\delta p_1 - \delta q_2 \left(-\frac{\partial^2 H}{\partial q \partial q}\delta q_1 - \frac{\partial^2 H}{\partial p \partial q}\delta p_1 \right) \boxed{= 0}$$

J is conserved

J: symplectic current in F-R

$\partial^\mu J_\mu = 0$ ~ independence of the covariant phase space symplectic form of Σ

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O fim duma viagem é apenas
o começo doutra

(J. Saramago)