GEOMETRIA II

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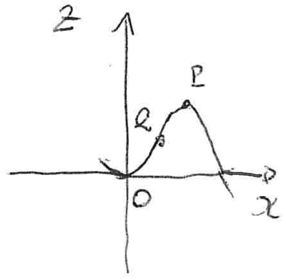
Prova scritta del 12 settembre 2019

① Sia data $\mathcal{C}: \ z = x^4(1-x)$, $x \in [0,1)$

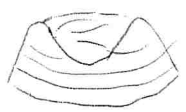
Si consideri la superficie Σ ottenuta ruotando $\mathcal{C}$ attorno all'asse z. Se ne calcoli la curvatura gaussiana in



$O: (0,0,0)$

$P: \left(\frac{4}{5}, 0, \frac{4^4}{5^5}\right)$

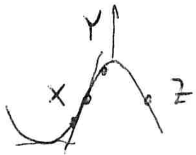
$Q: \left(\frac{3}{5}, 0, \frac{2 \cdot 3^4}{5^5}\right)$



Sugg.: $K(O)$: $z = \rho^4 - \rho^5 = \rho^4 + \ldots$

$K(P)$ calcolare k_n (parallelo per P)

$K(Q)$ = k_n (meridiano per $Q = \mathcal{C}$)



Determinare, per via geometrica, il segno di $K(X)$, $K(Y)$, $K(Z)$ in figura

② Sia data, su $\{(x,y) \mid x>0, y>0\}$ la metrica riemanniana

$$ds^2 = \frac{dx^2}{x^2} + e^{2y} dy^2$$

Se ne determinino la curvatura gaussiana e le geodetiche.

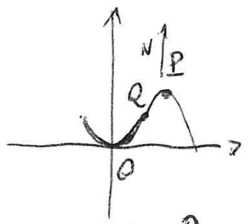
Tempo a disposizione: 1h 30m

Le risposte vanno adeguatamente giustificate

①

$\mathcal{C}: \quad z = x^4(1-x) \qquad x \in [0,1)$

Ruotandola attorno all'asse z si ottiene una sup di rotazione Σ :
$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = \rho^4(1-\rho) \end{cases}$$



$O: (0,0,0)$

$P: \left(\frac{4}{5}, 0, \frac{4^4}{5^5}\right)$

$Q = \left(\frac{3}{5}, 0, \frac{2\cdot 3^4}{5^5}\right)$

$' = \frac{d}{dx}$

$z = x^4 - x^5$

$z' = 4x^3 - 5x^4 = x^3(4-5x) = 0$ per $x = \frac{4}{5}$, $x = 0$ tripla

$z'' = 12x^2 - 20x^3 = x^2(12-20x) = 0$ per $x = \frac{\cancel{12}^{3}}{\cancel{20}_{5}} = \frac{3}{5}$ e $x=0$, doppia

$Q: \left(\frac{3}{5}, \frac{2\cdot 3^4}{5^5}\right)$

$z(P) = \left(\frac{4}{5}\right)^4\left(1-\frac{4}{5}\right) = \left(\frac{4}{5}\right)^4 \cdot \frac{1}{5} = \frac{4^4}{5^5}$

$z(Q) = \left(\frac{3}{5}\right)^4\left(1-\frac{3}{5}\right) = \left(\frac{3}{5}\right)^4 \frac{2}{5} = \frac{2\cdot 3^4}{5^5}$

Calcolare $K(O)$, $K(P)$, $K(Q)$

Sup. per O: McLaurin $z = \rho^4(1-\rho) = \rho^4 + \dots$

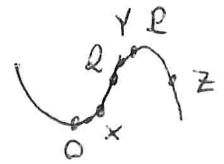
$\Rightarrow$ O punto planare

$K(P) = 0$: infatti $k_n(\text{parallelo per } P) = \frac{1}{\frac{4}{5}} \cdot \langle n, N\rangle = 0$

$K(Q) = 0$: infatti $k^{\mathcal{C}}(Q) = \frac{z''(Q)}{(1+z'(Q)^2)^{3/2}} = 0$ (flesso)

$\Rightarrow k_n(\text{merid}) = 0$

$K(X) > 0 \quad K(Y) < 0 \quad K(Z) > 0$



(2)

Su $\{(x,y) \mid x>0,\ y>0\}$ sia data la metrica $ds^2 = \frac{dx^2}{x^2} + e^{2y} dy^2$

Se ne determinino la curvatura gaussiana e le geodetiche.

Sol $\quad ds^2 = \left(\frac{dx}{x}\right)^2 + (e^y dy)^2$

$K=0$: calcolo diretto $\quad = (d \log x)^2 + (d e^y)^2$

$\begin{cases} \xi = \log x \\ \eta = e^y \end{cases} \qquad ds^2 = d\xi^2 + d\eta^2$ (metrica euclidea sul piano (ξ, η) $\xi \in \mathbb{R}$, $\eta > e$

geodetiche su (ξ,η): $\quad a\xi + b\eta + c = 0$ (rette) $\quad (a,b) \neq (0,0)$

$\Rightarrow \quad a \log x + b e^y + c = 0$