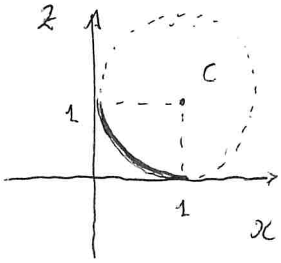
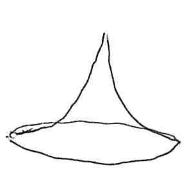
GEOMETRIA II

Prova scritta del 6/6/2019

(1)



Sia $\mathcal{C}$ l'arco di circonferenza in figura (centro C e raggio $R=1$). Si consideri la superficie Σ ottenuta ruotando $\mathcal{C}$ attorno all'asse z. Si calcoli la curvatura gaussiana K in un pto generico di Σ (è sufficiente scegliere P sulla curva profilo $\mathcal{C}$)



(2) Con riferimento all'esercizio 1, si calcolino la curvatura geodetica e la curvatura normale di un generico parallelo di Σ. Si tratta di geodetiche? È possibile stabilirlo a priori?

Tempo a disposizione 1h 30m

Le risposte vanno adeguatamente giustificate.

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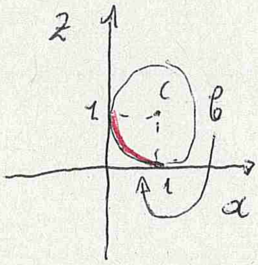
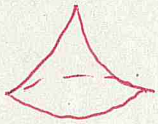


$C: (1,1) \quad R=1$

$$(x-1)^2+(z-1)^2=1 \qquad (z-1)^2 = 1-(x-1)^2$$

$$z-1 = \pm\sqrt{1-(x-1)^2}$$



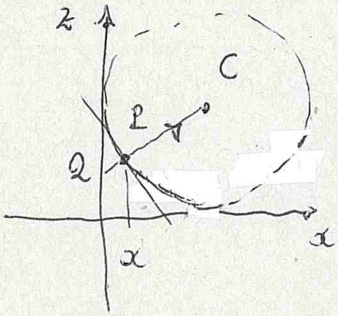
$$z = 1 \pm \sqrt{1-(x-1)^2}$$

$\mathscr{C} \longrightarrow$

$$z = 1-\sqrt{1-(x-1)^2} = 1-\sqrt{1-x^2+2x-1}$$
$$= 1-\sqrt{x(2-x)}$$

Calcoliamo K in un pto di $\mathscr{C}$ (ciò basta in generale)

Usiamo il teorema della gran normale



$$N = \overline{PQ} = |k_2|^{-1}$$

$C: (1,1)$

$P: (x,\ 1-\sqrt{x(2-x)})$

$Q =$ retta $CP \cap$ asse z (per le note proprietà della circ.)

$$\begin{cases} z-1 = \underbrace{\dfrac{-\sqrt{\ }}{x-1}}\cdot(X-1) & CP \\ X=0 \end{cases}$$

$m = \dfrac{\sqrt{\ }}{1-x}$

$CP:\ \dfrac{Z-z_C}{X-x_C} = \dfrac{z_P-z_C}{x_P-x_C}$

$$z = z_Q = \frac{\sqrt{\ }}{x-1}+1 = \frac{\sqrt{x(2-x)}}{x-1}+1$$

$P: (x,\ 1-\sqrt{\ })$

$Q: (0,\ \frac{\sqrt{\ }}{x-1}+1)$

$$\overline{PQ}^2 = x^2+\left(\frac{\sqrt{\ }}{x-1}+\sqrt{\ }\right)^2 = x^2+(\sqrt{\ })^2\left(\frac{1}{x-1}+1\right)^2$$

$$= x^2+x(2-x)\cdot\left(\frac{1+x-1}{x-1}\right)^2 = x^2+x(2-x)\frac{x^2}{(x-1)^2}$$

$$= \frac{x^2(x-1)^2+x^3(2-x)}{(x-1)^2} = \frac{x^2\left[(x-1)^2+x(2-x)\right]}{(x-1)^2}$$

PQ^2

$$= \frac{x^2}{(x-1)^2}\left[(x-1)^2 + x(2-x)\right] = \frac{x^2}{(x-1)^2}\left[\cancel{x^2} - \cancel{2x} + 1 + \cancel{2x} - \cancel{x^2}\right]$$

$$= \frac{x^2}{(x-1)^2}$$

$k_2 = \dfrac{x-1}{x} < 0$ per $x \in (0,1)$

OK

$k_1 =$ curv. meridiano $= 1$ (ovvio)

(ok il segno $+$) $\rightsquigarrow$ $N = n$

$K = k_1 k_2 = \dfrac{x-1}{x} < 0$ (da attendersi)

($k \to 0$ for $x \to 1$
$k \to \infty$ for $x \to 0$)

Variante

lavoriamo nel $P_0: (x_0, 0, z_0 = z(x_0))$

Cfr: $f(x,y) = 0$ $\quad (x-1)^2 + (z-1)^2 - 1 = 0$

norm in P_0: $f_z(P_0)(x - x_0) - f_x(P_0)(z - z_0) = 0$

$f_x = 2(x_0 - 1) \qquad f_z = 2(z_0 - 1)$

norm: $\rightarrow \begin{cases} (z_0 - 1)(x - x_0) - (x_0 - 1)(z - z_0) = 0 \\ x = 0 \end{cases}$

$(1 - z_0)x_0 - (x_0 - 1)(z - z_0) = 0$

$$z - z_0 = \frac{1 - z_0}{x_0 - 1} \cdot x_0 \Rightarrow z_Q = z_0 + \frac{x_0(1 - z_0)}{x_0 - 1}$$

$$= \frac{\cancel{z_0 x_0} - z_0 + x_0 - \cancel{x_0 z_0}}{x_0 - 1} = \frac{x_0 - z_0}{x_0 - 1}$$

$$P_0 = (x_0, 0, z_0)$$

$$Q = \left(0, 0, \frac{x_0 - z_0}{x_0 - 1}\right)$$

$$\overline{P_0Q}^2 = x_0^2 + \left(\frac{x_0 - z_0}{x_0 - 1} - z_0\right)^2 =$$

$$= x_0^2 + \frac{(x_0 - \cancel{z_0} - x_0 z_0 + \cancel{z_0})^2}{(x_0 - 1)^2}$$

$$= x_0^2 + \frac{x_0^2 (1 - z_0)^2}{(x_0 - 1)^2}$$

$$= x_0^2 \left[\frac{\overbrace{(x_0 - 1)^2 + (1 - z_0)^2}^{=1}}{(x_0 - 1)^2}\right] = \frac{x_0^2}{(x_0 - 1)^2}$$

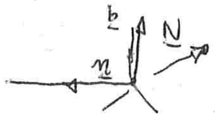
$$N = \frac{x_0}{1 - x_0} \qquad k_2 = -\frac{1}{N} = \frac{x_0 - 1}{x_0}$$

✓

2 bis

② Calcoliamo k_g per un generico parallelo

(x costante lungo il parallelo)



$$\underline{b} = (0,0,1)$$

$$k_g = k \cdot \langle \underline{b}, \underline{N} \rangle \qquad \underline{N} = \frac{1}{\|\ \|}\left(1,\ 0,\ \frac{\sqrt{x(2-x)}}{1-x}\right)$$

$$\|\cdot\| = \sqrt{1+\frac{x(2-x)}{(1-x)^2}} = \sqrt{\frac{(1-x)^2+x(2-x)}{(1-x)^2}} = \frac{\sqrt{1+x^2-\cancel{2x}+\cancel{2x}-x^2}}{1-x}\ (>0) = \frac{1}{1-x}$$

$k = \frac{1}{x}$ ← raggio del parallelo

$$k_g = \frac{1}{x}\cdot\frac{1}{\|\ \|}\,\frac{\sqrt{x(2-x)}}{1-x}$$

$$= \frac{1}{x}\cdot\frac{\sqrt{x(2-x)}}{\cancel{1-x}}\cdot(\cancel{1-x})$$

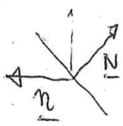
$$= \frac{\sqrt{x(2-x)}}{x} = \sqrt{\frac{2-x}{x}}$$

nessun parallelo è una geodetica
(chiaro a priori!)

$$k_n^2 + k_g^2 = k^2 = \frac{1}{x^2}$$

$$k_n = \pm\sqrt{k^2 - k_g^2}$$

si sceglie il segno "−" ⇝ $-\sqrt{\frac{1}{x^2} - k_g^2}$



$$= -\sqrt{\frac{1}{x^2} - \frac{x(2-x)}{x^2}} = -\sqrt{\frac{1-2x+x^2}{x^2}} = -\frac{1-x}{x}$$

Variante rapida:

$k_n = \frac{x-1}{x}$ ... già calcolata

$$k = \frac{1}{x} \qquad k_g^2 = \frac{1}{x^2} - \frac{(x-1)^2}{x^2} = \frac{1-(x-1)^2}{x^2}$$

$$= \frac{(\cancel{1}+x-\cancel{1})(1-x+1)}{x^2} = \frac{x(2-x)}{x^2} = \frac{2-x}{x} \quad ✓$$