GEOMETRIA II

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Prova scritta dell' 11 settembre 2014

(1) Sia Σ la superficie ottenuta ruotando

$$\mathcal{C}: \quad y = 1 - x^2 + ax \quad , \quad x \in (0,2) \qquad a \in \mathbb{R}$$

attorno all'asse x. Determinare a in modo che il parallelo corrispondente a $x=1$ risulti essere una geodetica. Si calcoli poi, lungo tale parallelo, la curvatura gaussiana di Σ.

(2) Data la curva spaziale $\mathcal{C}: \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$

Si determinino κ e τ, e si individuino i piani principali per $t = \frac{\pi}{3}$ (equazioni cartesiane).

Tempo a disposizione: 1h 30m

Le risposte vanno adeguatamente giustificate

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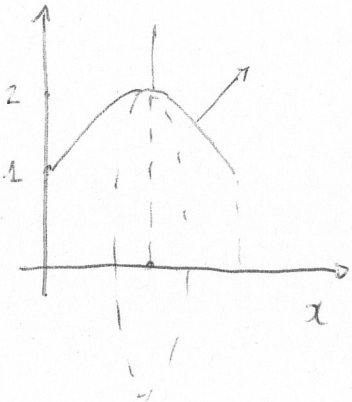

Sia Σ ottenuta ruotando

$\mathcal{C}: \quad y = 1 - x^2 + ax$

$x \in (0,2)$

attorno all' asse x e orientata in modo naturale

Determinare a in modo che il parallelo corrispondente a $x=1$ sia una geodetica.
Calcolare poi, lungo tale parallelo, la curvatura gaussiana.

Sol. $x=1$ deve risultare pto critico di $y = 1 - x^2 + ax$
$y' = 0$

pertanto da

$$y = 1 - x^2 + ax$$

$$y' = -2x + a \quad \Rightarrow \text{(pon. } x=1\text{)} \quad \boxed{a = 2}$$

$$y'' = -2 \quad \rightarrow \quad \text{curvatura di } \mathcal{C} = \text{meridiano}$$

$$y(1) = 1 - 1 + 2 = 2$$

Si trova, successivamente

notare

$$K = k_1 \cdot k_2 = (-2) \cdot \left(-\frac{1}{2}\right) = +1$$

curv(norm) meridiano

curv. normale del par = curv. del parallelo ($\langle N, n \rangle = -1$)

$K = +1$

(ragionevole...)

② Data $\mathscr{C}$: $\begin{cases} x = \cos t \\ y = \sin t \\ y = t \end{cases}$ (elica cilindrica)

determinarne k, τ e i piani principali in $t = \frac{\pi}{3}$

Sol.

$$\begin{array}{llll} \underline{r} : & (\cos t, \sin t, t) & & \text{per } t = \frac{\pi}{3} \quad \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right) \\ \dot{\underline{r}} : & (-\sin t, \cos t, 1) & & \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right) \\ \ddot{\underline{r}} : & (-\cos t, -\sin t, 0) & & \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right) \\ \dddot{\underline{r}} : & (\sin t, -\cos t, 0) & & \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right) \end{array}$$

$$k = \frac{\|\dot{\underline{r}} \times \ddot{\underline{r}}\|}{\|\dot{\underline{r}}\|^3} = \frac{2^{\frac{1}{2}}}{2^{3/2}} = 2^{\frac{1}{2}-\frac{3}{2}} = 2^{-1} = \frac{1}{2}$$

$\boxed{\dot{\underline{r}} \times \ddot{\underline{r}}}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \boxed{\underline{i} \sin t - \underline{j} \cos t + \underline{k}}$$

$\Rightarrow \| \quad \| = \sqrt{1+1} = \sqrt{2}$

$\|\dot{\underline{r}}\| = \sqrt{1+1} = \sqrt{2}$

$$\tau = -\frac{\langle \dot{\underline{r}} \times \ddot{\underline{r}}, \dddot{\underline{r}} \rangle}{\|\dot{\underline{r}} \times \ddot{\underline{r}}\|^2} = -\frac{\sin^2 t + \cos^2 t}{2} = -\frac{1}{2}$$

$$\boxed{\begin{array}{l} k = \frac{1}{2} \\ \tau = -\frac{1}{2} \end{array}}$$

piano normale

$$(x - \cos t)(-\sin t) + (y - \sin t)\cos t + (z - t) \cdot 1 = 0$$

$$\left(x - \frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(y - \frac{\sqrt{3}}{2}\right)\frac{1}{2} + z - \frac{\pi}{3} = 0$$

$$-\frac{\sqrt{3}}{2}x + \frac{1}{2}y + z + \cancel{\frac{\sqrt{3}}{4}} - \cancel{\frac{\sqrt{3}}{4}} - \frac{\pi}{3} = 0$$

6 $$\boxed{-\frac{\sqrt{3}}{2}x + \frac{1}{2}y + z - \frac{\pi}{3} = 0}$$ ← piano normale

$$-3\sqrt{3}\,x + 3y + 6z - 2\pi = 0$$

$$\boxed{3\sqrt{3}\,x - 3y - 6z + 2\pi = 0}$$

piano osculatore: $\underline{b} \propto \dot{\underline{r}} \times \ddot{\underline{r}} = \underline{i}\sin t - \underline{j}\cos t + \underline{k}$

si trova

$$\left(x - \frac{1}{2}\right)\frac{\sqrt{3}}{2} + \left(y - \frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) + \left(z - \frac{\pi}{3}\right)(+1) = 0$$

$$\frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + z - \frac{\pi}{3} = 0$$

$$\frac{\sqrt{3}}{2}x - \cancel{\frac{\sqrt{3}}{4}} - \frac{1}{2}y + \cancel{\frac{\sqrt{3}}{4}} + z - \frac{\pi}{3} = 0$$

$$\boxed{\frac{\sqrt{3}}{2}x - \frac{1}{2}y + z - \frac{\pi}{3} = 0}$$ ← piano osculatore

(controllo $-\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{2} + 1 = -\frac{3}{4} - \frac{1}{4} + 1 = 0$ ✓)

piano rettificante: la dir. ad esso perpendicolare ($\propto \underline{n}$)

è data da un vett. $\propto \underline{b} \times \underline{t} = \underline{n}$

se $\underline{a} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right)$

$\underline{b} = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1\right)$

$\underline{a} \times \underline{b} = (1, \sqrt{3}, 0)$

prendiamo: $$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & +1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & +1 \end{vmatrix} =$$

$$\underline{i}\underbrace{\left(\frac{1}{2} + \frac{1}{2}\right)}_{1} - \underline{j}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + \underline{k}\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)$$

$\underline{a} \times \underline{b} \perp \underline{a}$, $\perp \underline{b}$ ✓

$$= \underline{i} + \sqrt{3}\,\underline{j}$$

$$\boxed{\left(x - \frac{1}{2}\right) + \sqrt{3}\left(y - \frac{\sqrt{3}}{2}\right) = 0}$$ ← piano rettificante