GEOMETRIA II

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(1) Sia data la superficie Σ descritta da

$$\underline{r}(u,v) = (e^u \cos v,\ e^u \sin v,\ u) \qquad u \in \mathbb{R},\ v \in [0, 2\pi)$$

Di che tipo di superficie si tratta?

Calcolarne la prima e la seconda forma fondamentale nonché, in due modi, la curvatura gaussiana.

Determinare le curvature principali.

(2) Sia Σ data come in (1).

Determinare l'operatore di forma in $P: (1, 0, 0)$ nonché l'indicatrice di Dupin e le direzioni principali e asintotiche.

Tempo a disposizione: 1h. 30

Le risposte vanno adeguatamente giustificate

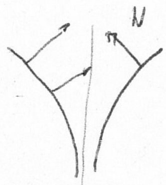
① $\Sigma: \ \underline{r}(u,v) = \left(e^u \cos v, \ e^u \sin v, \ u \right)$ $\qquad u \in \mathbb{R}, \quad v \in [0, 2\pi)$

$$\begin{cases} \underline{r}_u = \left(e^u \cos v, \ e^u \sin v, \ 1 \right) \\ \underline{r}_v = \left(-e^u \sin v, \ e^u \cos v, \ 0 \right) \end{cases}$$

è una sup. di rotazione

a priori $F = f = 0$

$$\begin{cases} \underline{r}_{uu} = \left(e^u \cos v, \ e^u \sin v, \ 0 \right) \\ \underline{r}_{uv} = \left(-e^u \sin v, \ e^u \cos v, \ 0 \right) \\ \underline{r}_{vv} = \left(-e^u \cos v, \ -e^u \sin v, \ 0 \right) \end{cases}$$



⚠

$$\boxed{\underline{r}_u \times \underline{r}_v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ e^u \cos v & e^u \sin v & 1 \\ -e^u \sin v & e^u \cos v & 0 \end{vmatrix} =$$

$$\|\underline{r}_u \times \underline{r}_v\| = \sqrt{e^{2u} + e^{4u}} = e^u \sqrt{1 + e^{2u}}$$

$$= \underline{i} \left(-e^u \cos v \right) - \underline{j} \left(e^u \sin v \right) + \underline{k} \ e^{2u}$$

$$= \left(-e^u \cos v \right) \underline{i} - \left(e^u \sin v \right) \underline{j} + e^{2u} \underline{k}$$

1^a f.f. (metrica) $\quad E = \|\underline{r}_u\|^2 = e^{2u} + 1$

$$\boxed{F = 0}$$

$$G = e^{2u}$$

$$\underline{N} = \left(-\frac{\cos v}{\sqrt{1+e^{2u}}}, \ -\frac{\sin v}{\sqrt{1+e^{2u}}}, \ \frac{e^u}{\sqrt{1+e^{2u}}} \right)$$

$$e = -\frac{e^u}{\sqrt{\ }} \cos^2 v - \frac{e^u}{\sqrt{\ }} \sin^2 v = -\frac{e^u}{\sqrt{1+e^{2u}}}$$

$$\boxed{f = \ldots 0}$$

$$g = +\frac{e^u}{\sqrt{\ }} \cos^2 v + \frac{e^u}{\sqrt{\ }} \sin^2 v = \frac{e^u}{\sqrt{1+e^{2u}}}$$

$$K = \frac{eg}{EG} = -\frac{\cancel{e^{2u}}}{(1+e^{2u})\cdot \cancel{e^{2u}}(1+e^{2u})} = -\frac{1}{(1+e^{2u})^2}$$

* Calcolo alternativo di K

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\} \qquad \begin{aligned} E &= e^{2u}+1 \\ G &= e^{2u} \end{aligned}$$

$$E_v = 0 \qquad G_u = 2e^{2u}$$

$$\frac{G_u}{\sqrt{EG}} = \frac{2e^{2u}}{\sqrt{e^{2u}+1}\sqrt{e^{2u}}} = \frac{2e^{2u}}{(e^{2u}+1)^{\frac{1}{2}}e^{u}} = \frac{2e^{u}}{(e^{2u}+1)^{\frac{1}{2}}}$$

$$\frac{\partial}{\partial u}(\ ) = \frac{2e^{u}(e^{2u}+1)^{\frac{1}{2}} - 2e^{u}\cdot\frac{1}{2}(e^{2u}+1)^{-\frac{1}{2}}\cdot 2e^{2u}}{e^{2u}+1}$$

$$= \frac{2e^{u}(e^{2u}+1) - 2e^{u}e^{2u}}{(e^{2u}+1)^{\frac{3}{2}}} =$$

$$= \frac{2e^{u}(e^{2u}+1-e^{2u})}{(e^{2u}+1)^{\frac{3}{2}}} = \frac{2e^{u}}{(e^{2u}+1)^{\frac{3}{2}}}$$

$$K = -\frac{1}{\cancel{2}}\frac{1}{(e^{2u}+1)^{\frac{1}{2}}\cancel{e^{u}}}\cdot\frac{\cancel{2}\cancel{e^{u}}}{(e^{2u}+1)^{\frac{3}{2}}} = -\frac{1}{(e^{2u}+1)^2} \quad \checkmark$$

$$H = \frac{1}{2}\left(\frac{e}{E} + \frac{g}{G}\right) = \frac{1}{2}\,\frac{eG + gE}{EG}$$

$$\frac{e}{E} = k_1 \qquad \frac{g}{G} = k_2$$

$$k_1 = -\frac{e^{u}}{\sqrt{1+e^{2u}}}\,\frac{1}{(e^{2u}+1)} = \frac{-e^{u}}{(e^{2u}+1)^{\frac{3}{2}}}$$

$$k_2 = \frac{e^{u}}{\sqrt{1+e^{2u}}}\cdot\frac{1}{e^{2u}} = \frac{1}{(1+e^{2u})^{\frac{1}{2}}e^{u}}$$

$$= \frac{1}{2}\;\frac{-\frac{e^{u}}{\sqrt{\;}}\,e^{2u} + \frac{e^{u}}{\sqrt{\;}}\,(e^{2u}+1)}{(e^{2u}+1)\,e^{2u}}$$

$$= \frac{1}{2}\cdot\frac{1}{(e^{2u}+1)\,e^{2u}}$$

② Operatore di forma per $u=0,\ v=0$

$P: (1, 0, 0)$

$$\underset{(r_u, r_v)}{m_{ee}(S)} = \begin{pmatrix} \frac{e}{E} & 0 \\ 0 & \frac{g}{G} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (H>0)$$

Direzioni asintotiche: $\quad e\dot{u}^2 + 2f\dot{u}\dot{v} + g\dot{v}^2 = 0$

qui $\quad e\dot{u}^2 + g\dot{v}^2 = 0$

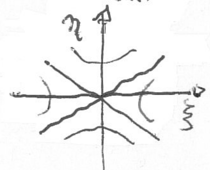
$$-\frac{1}{\sqrt{2}}\dot{u}^2 + \frac{1}{\sqrt{2}}\dot{v}^2 = 0$$

$$\dot{u}^2 - \dot{v}^2 = 0 \qquad \dot{u} = \pm\dot{v}$$

Dupin (in P)

$k_1\xi^2 + k_2\eta^2 = \pm 1$

$\underbrace{-\frac{1}{2\sqrt{2}}\xi^2 + \frac{1}{2}\eta^2}_{(*)} = \pm 1$



asintoti: $(*) = 0$

$r_u = (1, 0, 1)$ $\qquad$ dir asintotiche

$r_v = (0, 1, 0)$ $\qquad a_{1,2} = r_u \pm r_v$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} \pm \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\\pm 1\\1\end{pmatrix}$$

rette:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\\pm 1\\1\end{pmatrix} \Rightarrow \begin{cases} x = 1+t \\ y = \pm t \\ z = t \end{cases}$$

dir. principali $\rightsquigarrow r_u$ e r_v

rette corr:

$$\begin{cases} \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\1\end{pmatrix} & \begin{cases} x = 1+t \\ y = 0 \\ z = t \end{cases} \\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\0\end{pmatrix} & \begin{cases} x = 1 \\ y = t \\ z = 0 \end{cases} \end{cases}$$