GEOMETRIA II

a.a. 2014/15

Prof. M. Spera, UCSC - Brescia

Prova scritta del 10/9/2015

(1) Data la curva $\mathscr{C}: \begin{cases} x^2+y^2+z^2=1 \\ 2x^2-y^2=1 \end{cases}$

e $P: \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \in \mathscr{C}$

Dopo aver verificato che $\mathscr{C}$ è regolare, calcolare $\kappa(P)$ (e $\tau(P)$, facol.) [si utilizzi il calcolo implicito]

(2) Data Σ : $\frac{x^2}{2}+y^2-z^2-1=0$, orientata con la normale "esterna" in $P:(0,1,0)$ $(\in \Sigma)$ determinarne le due forme fondamentali, la curvatura gaussiana, la curvatura media, l'operatore di forma e le relative direzioni principali e asintotiche.

[localmente si scriva $y = y(\underset{1^a}{z}, \underset{2^a}{x})$ e si utilizzi il calcolo implicito]

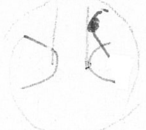
Tempo a disposizione: 1h.30m

Le risposte vanno adeguatamente giustificate.

(1) $\mathcal{C}: \begin{cases} x^2+y^2+z^2=1 \\ 2x^2-y^2=1 \end{cases}$ $\leftarrow$ sfera, $\leftarrow$ cilindro iperbolico

$(x>0)$

Sia $P: \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \in \mathcal{C}$



Calcolare k e τ in P

Utilizziamo il calcolo implicito

$\mathcal{C}$ è regolare (v. oltre)

$x = x(s)$ ecc.

$s=0 \leftrightarrow P$

$$\begin{cases} x^2+y^2+z^2=1 \\ 2x^2-y^2=1 \end{cases}$$

vale sempre!

$\Downarrow$

(7) $x'^2+y'^2+z'^2=1$

(8) $\cancel{2}x'x''+\cancel{2}y'y''+\cancel{2}z'z''=0$

(9) $x''^2+y''^2+z''^2+x'x'''+y'y'''+z'z'''=0$

(1) $\cancel{2}xx'+\cancel{2}yy'+\cancel{2}zz'=0$

(2) $\overset{2}{\cancel{4}}xx'-\cancel{2}yy'=0$

(3) $x'^2+xx''+y'^2+yy''+z'^2+zz''=0$

(4) $2x'^2+2xx''-y'^2-yy''=0$

$\to$ (3) $1+xx''+yy''+zz''=0$ per (7)

$\to$ (4) $2x'^2+2xx''-y'^2-yy''=0$

(5) $x'x''+y'y''+z'z''+xx'''+yy'''+zz'''=0$

(6) $\underbrace{4x'x''+2x'x''}_{6x'x''}+2xx'''\underbrace{-2y'y''-y'y''}_{-3y'y''}-yy'''=0$

(1): $\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}z'=0$

(2): $\frac{2}{\sqrt{2}}x'=0$ $\Rightarrow$ $\boxed{x'=z'=0}$

(7) $y'^2=1 \Rightarrow y'=\pm 1$

scegliamo $\boxed{y'=+1}$

$\boxed{\mathbf{r}'=(0,1,0)}$

(3) $1+\frac{1}{\sqrt{2}}x''+\frac{1}{\sqrt{2}}z''=0$

(4) $2\frac{1}{\sqrt{2}}x''-1=0 \to \sqrt{2}x''=1 \Rightarrow x''=\frac{1}{\sqrt{2}}$

$1+\frac{1}{2}+\frac{1}{\sqrt{2}}z''=0$ $\quad \frac{3}{2}+\frac{1}{\sqrt{2}}z''=0$ $\quad z''=-\frac{3}{2}\cdot\sqrt{2}$

(8) $y''=0$ $\quad z''=-\frac{3}{2}\sqrt{2}$ $\longrightarrow$ $\boxed{\mathbf{r}''=\left(\frac{1}{\sqrt{2}}, 0, -\frac{3}{2}\sqrt{2}\right)}$

5 $\frac{1}{\sqrt{2}} x''' + \frac{1}{\sqrt{2}} z''' = 0$

6 $\frac{2}{\sqrt{2}} x''' = 0 \qquad x''' = z''' = 0$

$$\boxed{r''' : (0, -5, 0)}$$

9

$$\frac{1}{2} + \frac{9}{2} + y''' = 0$$

$$y''' = -5$$

$$\boxed{\begin{array}{l} k = \sqrt{5} \\ \tau = 0 \end{array}}$$

$$k = \| r'' \| = \sqrt{\frac{1}{2} + \frac{9}{2}} = \sqrt{5}$$

$$\tau = - \frac{\langle r' \times r'', r''' \rangle}{k^2} = - \frac{\begin{vmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{3}{\sqrt{2}} \\ 0 & -5 & 0 \end{vmatrix}}{k^2} = 0$$

da attendersi per simmetria

4 regolarità $\nabla f \times \nabla g \neq 0$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2x & 2y & 2z \\ 4x & -y & 0 \end{vmatrix} = 0 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ 2x & -y & 0 \end{vmatrix} = \underline{i}\,(+yz) - \underline{j}\,(-2xz) + \underline{k}\,(-3xy) = 0$$

$$\Rightarrow \begin{cases} xy = 0 \\ yz = 0 \\ xz = 0 \\ x^2+y^2+z^2=1 \\ 2x^2-y^2=1 \end{cases} \Rightarrow x = 0 \lor y = 0$$

se $x = 0$ e $y \neq 0$, z , $-y^2 = 1$ assurda.

$x = 0$ e $y = 0$ assurdo $0 = 1$

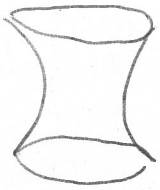
$x \neq 0 \Rightarrow y = 0$

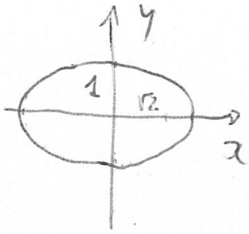
$\Rightarrow x = \pm \frac{1}{\sqrt{2}} \qquad \frac{1}{2} + z^2 = 1$

$z = \pm \frac{1}{\sqrt{2}}$

$\Rightarrow xz \neq 0$ NO

(2)





$$\frac{x^2}{2} + y^2 = 1$$

$$x^2 + 2y^2 = 2$$

$$\Sigma : \quad f = \frac{x^2}{2} + y^2 - z^2 - 1 = 0 \quad \text{iperboloide iperbolico}$$
(rigato)

$P : (0, 1, 0)$ $\qquad \frac{\partial f}{\partial y}(P) = 2y(P) = 2 \neq 0$

Dini! $\quad y = y(z, x)$

$$\mathcal{F}(z,x) = f(x, y(z,x), z) \equiv 0$$

$$f \quad \frac{x^2}{2} + y^2 - z^2 - 1 = 0$$

$$\mathcal{F}_x = x + 2y\,y_x = 0 \qquad \boxed{y_x(P) = 0} \text{ (chiaro)}$$

$$\mathcal{F}_z = 2y\,y_z - 2z = 0 \qquad \boxed{y_z(P) = 0}$$

$$\mathcal{F}_{xx} = 1 + 2y_x^2 + 2y\,y_{xx} = 0$$

$$\mathcal{F}_{xz} = 2y_z y_x + 2y\,y_{xz} = 0$$

$$\mathcal{F}_{zz} = y_z^2 + y\,y_{zz} - 1 = 0$$

in P:

$$1 + 2y_{xx} = 0 \qquad \boxed{y_{xx}(P) = -\tfrac{1}{2}}$$

$$2y_{xz} = 0 \qquad \boxed{y_{xz}(P) = 0}$$

$$y_{zz} - 1 = 0 \qquad \boxed{y_{zz} = 1}$$

$$\underline{r}(z,x) = (x, y(z,x), z)$$

$$\underline{r}_z = (0, y_z, 1)$$

$$\underline{r}_x = (1, y_x, 0)$$

in P

$$\underline{r}_z^0 = (0, 0, 1) = \underline{k}$$

$$\underline{r}_x^0 = (1, 0, 0) = \underline{i}$$

$$\underline{r}_z^0 \times \underline{r}_x^0 = \underline{k} \times \underline{i} = \underline{j} = (0, 1, 0)$$

chiaro!

proseguiamo $\qquad \boxed{\underline{N}^0 = (0,1,0)}$

$$\underline{r}_{zz} = (0, y_{zz}, 0)$$

$$\underline{r}_{zx} = (0, y_{zx}, 0)$$

$$\underline{r}_{xx} = (0, y_{xx}, 0)$$

$$\boxed{\begin{aligned} \underline{r}_{zz}^0 &= (0, 1, 0) \\ \underline{r}_{zx}^0 &= (0, 0, 0) \\ \underline{r}_{xx}^0 &= (0, -\tfrac{1}{2}, 0) \end{aligned}}$$

I^a Forma fondamentale in P $\qquad$ II^a f. fond. in P

$$E = G = 1$$
$$F = 0$$

$$e = \langle \underline{r}^o_{zz}, \underline{N} \rangle = 1$$
$$f = \langle \underline{r}^o_{zx}, \underline{N} \rangle = 0$$
$$g = \langle \underline{r}^o_{xx}, \underline{N} \rangle = -\frac{1}{2}$$

~ matrice hessiana di y

$$K = \frac{eg - f^2}{EG - F^2} = 1\left(-\frac{1}{2}\right) = -\frac{1}{2}$$ come era da attendersi

$$H = \frac{1}{2}\,\frac{eG - 2fF + Eg}{EG - F^2} = \frac{1}{2}\Big(\underbrace{1 - \frac{1}{2}}_{\frac{1}{2}}\Big) = \frac{1}{4}$$ $\quad K = k_1$

(k_1 e k_2 : radici di $\quad k^2 - 2Hk + K = 0$

$$k^2 - \frac{1}{2}k - \frac{1}{2} = 0 \qquad 2k^2 - k - 1 = 0$$

$$\left(k = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4} \begin{matrix} -\frac{1}{2} \\ 1 \end{matrix} \right)$$

$$\mathcal{M}(S) = \begin{pmatrix} \frac{e}{E} & \frac{f}{E} \\ \frac{f}{G} & \frac{g}{G} \end{pmatrix}$$ se $F = 0$

direzioni principali
$\begin{cases} x = 0 \\ y = 1 \end{cases}$
$\begin{cases} z = 0 \\ y = 1 \end{cases}$

$$\mathcal{M}(S) = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Dupin

$$\xi^2 - \frac{1}{2}\eta^2 = \pm 1$$

direzioni asintotiche: $\quad e\dot{u}^2 + 2f\dot{u}\dot{v} + g\dot{v}^2 = 0 \qquad u = z,\ v = x$

$$\dot{u}^2 - \frac{1}{2}\dot{v}^2 = 0 \qquad \alpha^2 - \frac{1}{2}\beta^2 = 0$$

$$\beta^2 = 2\alpha^2$$
$$\beta = \pm\sqrt{2}\,\alpha$$

$$\underline{v}_1 = \alpha\,\underline{r}_u + \beta\,\underline{r}_v$$
$$= \alpha\,\underline{r}_z + \beta\,\underline{r}_x$$
$$= \alpha\,\underline{k} \pm \sqrt{2}\,\alpha\,\underline{i} = \alpha(\pm\sqrt{2}, 0, 1)$$

∄ rette (contenute in Σ) $\qquad \underline{r} = \underline{r}_P + t\,\underline{v}_{1/2}$

$$\begin{cases} x = \pm t\sqrt{2} \\ y = 1 \\ z = t \end{cases}$$