GEOMETRIA II

a.a. 2014/15

Prof. M. Spera, UCSC-Brescia

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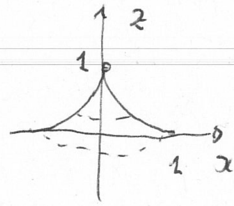
(1) Sia data, nel piano (x,z), $\mathcal{C}: \ z=(x-1)^2$, $0<x<1$; si consideri la superficie Σ ottenuta ruotando $\mathcal{C}$ attorno all'asse z. Si determini la curvatura gaussiana di Σ.
(Si può procedere in più modi)



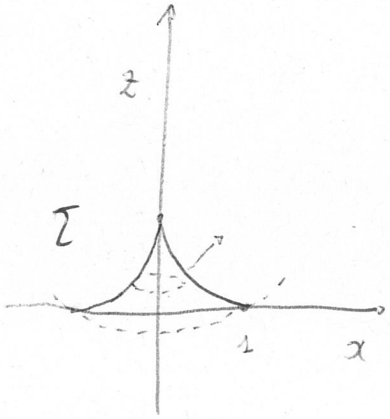
(2) Con riferimento all'es. 1, si consideri $\mathcal{C}_1 = \Sigma \cap \{z=\frac{1}{2}\}$. Si calcoli la curvatura geodetica di $\mathcal{C}_1$ (si può procedere in più modi).

(3) Fac. Determinare l'angolo di parallelismo ottenuto trasportando parallelamente il versore tangente in un pto del parallelo $\mathcal{C}_1$ lungo il parallelo stesso.

Tempo a disposizione: 1h 30m

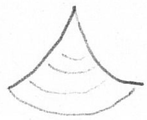
Le risposte vanno adeguatamente giustificate.

1



$z=(x-1)^2$

$0<\rho<1$



$$\begin{cases} x = \rho\cos\varphi \\ y = \rho \sin\varphi \\ z = (\rho-1)^2 \end{cases}$$ ok

k_1 : curvatura del meridiano

$z=(x-1)^2$

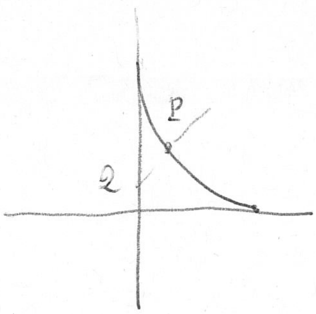
$z'=2(x-1)$

$z''=2$

$$k_1 = \frac{z''}{(1+z'^2)^{3/2}} = \frac{2}{(1+4(x-1)^2)^{3/2}}$$

$$k_1 = \frac{2}{(1+4(\rho-1)^2)^{3/2}}$$

(segno ok)



seminormale:

normale a $\mathcal{C}$ in P_0

$$\begin{cases} z - z_0 = -\dfrac{1}{z'_0}(x-x_0) \\ x = 0 \end{cases}$$

asse y

$$z_Q - z_P = -\frac{1}{z'_P}(\underset{x}{\overset{\shortparallel}{-x_P}}) = \frac{x_P}{z'_P} = \frac{x}{2(x-1)}$$

$$\overline{PQ} = \sqrt{x_P^2 + (z_Q - z_P)^2} = \sqrt{x^2 + \frac{1}{4(x-1)^2}x^2}$$

$$= \frac{x}{2(1-x)}\sqrt{4(x-1)^2+1}$$

(ok, $x>0$)

attenzione!

$$k_2 = -\frac{1}{\overline{PQ}} = -\frac{2(1-\rho)}{\rho}\,\frac{1}{\sqrt{4(\rho-1)^2+1}}$$

$$\Rightarrow K = k_1 k_2 = -\frac{2(1-\rho)}{\rho}\,\frac{1}{\sqrt{4(\rho-1)^2+1}}\cdot\frac{2}{(1+4(\rho-1)^2)^{3/2}}$$

$$= \frac{-4(1-\rho)}{\rho\,(4(\rho-1)^2+1)^2}$$

(< 0 ∴ T è a pti iperbolici) $\quad \frac{3}{2}+\frac{1}{2}=2$

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(2) Calcolare k_n e k_g di $\mathcal{C} = \Sigma \cap \{z=\frac{1}{2}\}$

1° $\quad k = \frac{1}{r} = \frac{\sqrt{2}}{\sqrt{2}-1}$

$\frac{1}{2} = (x-1)^2 \qquad x>0$

$1-x = \frac{1}{\sqrt{2}}$

$x = 1-\frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

$$k_n = -\frac{1}{\overline{PQ}} = -\frac{2(1-\rho)}{\rho}\,\frac{1}{\sqrt{4(\rho-1)^2+1}}$$

$\rho = \frac{\sqrt{2}-1}{\sqrt{2}}$

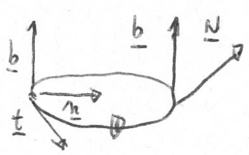
$1-\rho = \frac{1}{\sqrt{2}}$

$1-\frac{\sqrt{2}-1}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{2}+1}{\sqrt{2}}$

$$= -\frac{2\,\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}}\,\frac{1}{\sqrt{4\left(-\frac{1}{\sqrt{2}}\right)^2+1}}$$

$$= -\frac{2}{\sqrt{2}-1}\,\frac{1}{\sqrt{2+1}} = -\frac{2}{(\sqrt{2}-1)\sqrt{3}}$$

$$k_g = +\sqrt{k^2-k_n^2} = \sqrt{\frac{2}{(\sqrt{2}-1)^2}-\frac{4}{(\sqrt{2}-1)^2\cdot 3}} = \frac{1}{(\sqrt{2}-1)}\sqrt{2-\frac{4}{3}}$$



$$= \frac{1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}}{\sqrt{3}}$$

Variante $\quad k_g = k \cdot \langle \underline{b}, \underline{N} \rangle$

$$\underline{b} = (0,0,1)$$

$$\underline{N} = \frac{1}{r}\Big(1, 0, \underbrace{-\frac{1}{2(\rho-1)}}\Big)$$

$$\parallel$$

$$\frac{1}{\sqrt{1+\frac{1}{4(1-\rho)^2}}}\left(1, 0, \frac{1}{2(1-\rho)}\right)$$

$$1-\rho = \frac{1}{\sqrt{2}}$$

$$k_g = \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\frac{1}{2\frac{1}{\sqrt{2}}}}{\sqrt{1+\frac{1}{4\cdot\frac{1}{2}}}} =$$

$$= \frac{\sqrt{2}}{\sqrt{2}-1}\,\frac{\frac{1}{\sqrt{2}}}{\sqrt{1+\frac{1}{2}}} = \frac{1}{(\sqrt{2}-1)\frac{\sqrt{3}}{\sqrt{2}}} = \frac{\sqrt{2}}{(\sqrt{2}-1)\sqrt{3}} \quad \checkmark$$

Si noti altresì che: $\frac{\sqrt{2}}{(\sqrt{2}-1)}\frac{1}{\sqrt{3}} = \frac{\sqrt{2}(\sqrt{2}+1)}{(2-1)\sqrt{3}} = \frac{2+\sqrt{2}}{\sqrt{3}}$

3 $\quad$ Si ha $\quad \alpha_{\parallel} = 2\pi - \int_{\mathcal{C}_1} k_g \, ds \qquad ds = R\cdot d\varphi$

$$R = \frac{\sqrt{2}-1}{\sqrt{2}}$$

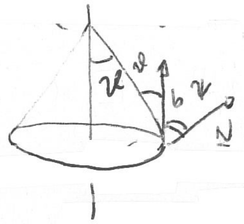
$$2\pi - \int_{\mathcal{C}_1} k_g \, ds = 2\pi - \cancel{R}\int_0^{2\pi} \frac{1}{\cancel{R}}\cdot\langle \underline{b}, \underline{N}\rangle \, d\varphi$$

$$= 2\pi\left[1 - \langle \underline{b}, \underline{N}\rangle\right] = 2\pi\cdot\left(1-\frac{1}{\sqrt{3}}\right)$$

$$= 2\pi\,\frac{\sqrt{3}-1}{\sqrt{3}}$$

$$\boxed{\frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{3}{2}}} = \frac{1}{\sqrt{3}} \;\; \parallel \;\; \langle \underline{b}, \underline{N}\rangle}$$

Si noti che la formula è in accordo con
$\alpha_{\parallel} = 2\pi(1-\sin\vartheta)$ (v. figura accanto)



Infatti $\langle \underline{b}, \underline{N}\rangle = \cos\varphi = \sin\vartheta$