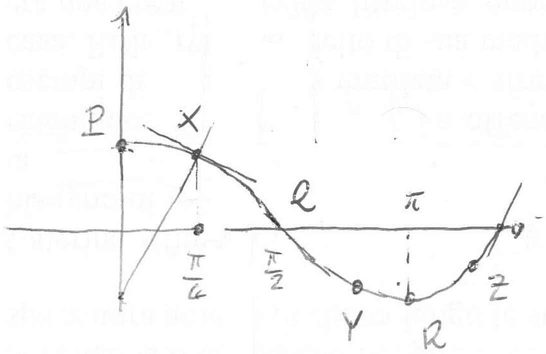
GEOMETRIA II

a.a. 2015/16

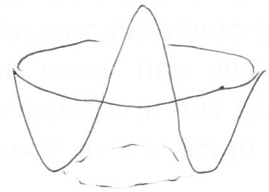
Prova scritta del 29 settembre 2016

① Sia data $\mathcal{C}$: $z = \cos x$, $x \in [0, \frac{3}{2}\pi]$

Sia Σ la sup. ottenuta ruotando $\mathcal{C}$ attorno all'asse z



"sombrero"



Si calcoli la curvatura gaussiana di Σ in P, Q, R.

[sugg. per P: si sviluppi $z = \cos \rho$ in serie di Mc-Laurin ...]

② Si calcoli la curvatura di $\mathcal{C}$ in $X = (\frac{\pi}{4}, 0, \frac{1}{\sqrt{2}})$ e, successivamente, la curvatura gaussiana di Σ in X.

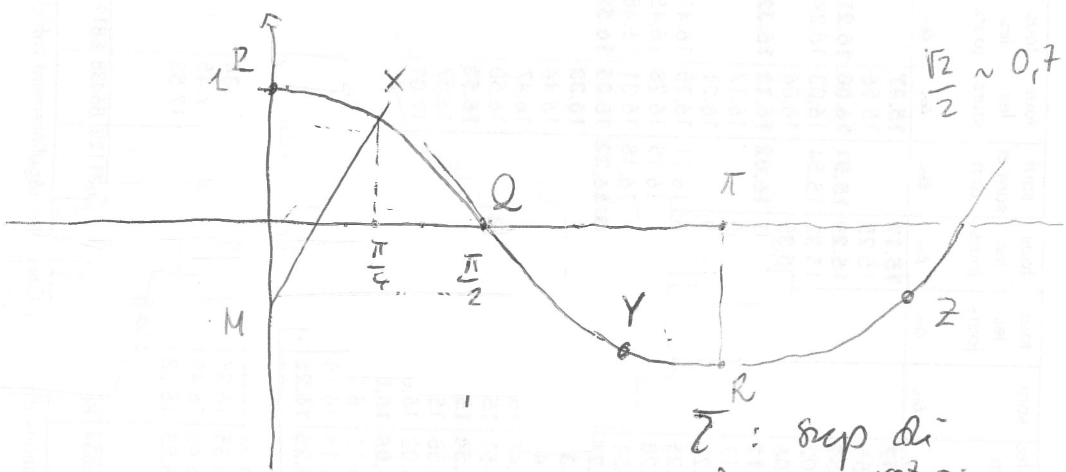
Qual è il segno di K in Y? E in Z?
(ci si riferisca alla figura)

Tempo a disposizione: 1h 30m

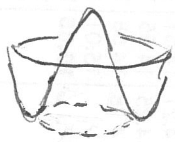
Le risposte vanno adeguatamente giustificate.

(1)

$z = \cos x$



Σ : sup di rotazione di $\mathcal{C}$: $z = \cos x$, $x \in [0, \frac{3}{2}\pi]$



Calcolare K in P, Q, R

K(P).

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = \cos \rho \end{cases} \qquad \rho = \sqrt{x^2+y^2}$$

$$z = \cos \rho = 1 - \frac{\rho^2}{2} + \dots$$

$$1 - \frac{x^2+y^2}{2} + \dots$$

$$\Rightarrow \mathcal{H} = \det \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1$$

(Hessiano in (0,0))

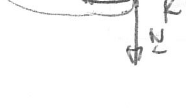
$$\Rightarrow K(P) = +1$$

in Q $\mathcal{C}$ (meridiano) ha un pto di flesso, siccome $k(Q) = 0$ (curvatura di $\mathcal{C}$)

$$\Rightarrow K(Q) = 0$$

In R la curvatura normale del parallelo è nulla:

$$\Rightarrow K(R) = 0$$



(2) calcoliamo $K(X)$ $\quad X:\left(\frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}\right)$

tangente a $\mathcal{C}$ in X. $\quad z-\frac{\sqrt{2}}{2} = \cos'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$

$\cos'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$$z-\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) \qquad \left(-\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}} = m\right)$$

normale: $\quad z-\frac{\sqrt{2}}{2} = +\sqrt{2}\left(x-\frac{\pi}{4}\right)$

int. con l'asse z: $\quad M: \begin{cases} z-\frac{\sqrt{2}}{2} = \sqrt{2}\left(x-\frac{\pi}{4}\right) \\ x = 0 \end{cases}$

$$z-\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}\pi \qquad z = \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4}\pi = \sqrt{2}\left(\frac{1}{2}-\frac{\pi}{4}\right) = \sqrt{2}\,\frac{2-\pi}{4} \quad (<0)$$

$X = \left(\frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}\right)$

$M = \left(0, 0, \frac{\sqrt{2}}{2}\cdot\frac{2-\pi}{2}\right)$

granormale: $N = \overline{XM} = \sqrt{\left(\frac{\pi}{4}\right)^2 + \frac{1}{2}\cdot\left(\frac{2-\pi}{2}-1\right)^2} = \sqrt{\frac{\pi^2}{4^2}+\frac{1}{2}\cdot\frac{\pi^2}{4}}$ $\quad \left(\cancel{1}-\frac{\pi}{2}-\cancel{1}\right)$

$$= \sqrt{\frac{\pi^2}{16}+\frac{\pi^2}{8}} = \pi\sqrt{\frac{1+2}{16}} = \sqrt{3}\,\frac{\pi}{4} \Rightarrow \boxed{N = \frac{\sqrt{3}}{4}\pi}$$

$k_2 = -\frac{1}{N}$

↑ curv. norm par

$k_1 =$ curv. merz.

$z = \cos x$

$z' = -\sin x$

$z'' = -\cos x$

$$k_1 = \frac{-\cos x}{(1+(-\sin x)^2)^{3/2}} = \frac{-\cos x}{(1+\sin^2 x)^{3/2}}$$

$$k_1(x) = \frac{-\cos\frac{\pi}{4}}{(1+\sin^2\frac{\pi}{4})^{3/2}} = \frac{-\frac{\sqrt{2}}{2}}{(1+\frac{1}{2})^{3/2}} = -\frac{\sqrt{2}}{2}\frac{1}{(3/2)^{3/2}}$$

$(3/2)^{3/2} = \sqrt{\frac{27}{8}}$

$$= -\frac{\sqrt{2}}{2}\frac{1}{\sqrt{\frac{27}{8}}} = -\frac{1}{\sqrt{2}}\frac{1}{\frac{\sqrt{27}}{2\sqrt{2}}}$$

$$= -\frac{2}{\sqrt{27}}$$

$27 = 3^3$

$27 \cdot 3 = 3^4$

$$\Rightarrow K(x) = \left(-\frac{1}{N}\right)\left(-\frac{2}{\sqrt{27}}\right) = \frac{2}{N\sqrt{27}} = \frac{4}{\sqrt{3}\pi}\cdot\frac{2}{\sqrt{27}} = \frac{8}{9\pi}$$

(hi...) > 0

in Y ī $K(Y) < 0$

in Z ī $K(Z) > 0$

dalla geom. del problema.