GEOMETRIA II — a.a. 2016/17

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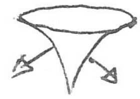
Prova scritta del $\boxed{\text{14 Settembre 2017}}$

Si orienti Σ così:



(1) Sia data, sul piano (x,z), la curva

$$\mathcal{C}: \quad z = x^{\frac{1}{2}}, \quad x \in (0,2)$$

Sia Σ la superficie ottenuta ruotando $\mathcal{C}$ attorno all'asse z. Si determinino in due modi le curvature principali e la curvatura gaussiana di Σ nel punto $P: (1,0,1)$

(2) Con riferimento all'esercizio precedente, si determini, in $P: (1,0,1)$, l'operatore di forma di Σ e si individuino le direzioni principali e le direzioni asintotiche su $T_P\Sigma$.

Tempo a disposizione: 1h30m

Le risposte vanno adeguatamente giustificate.

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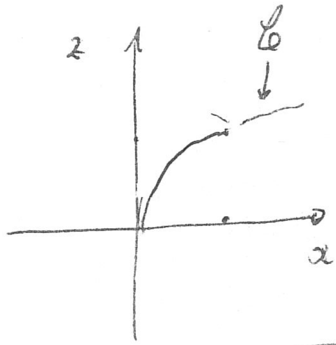
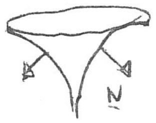


$z = x^{\frac{1}{2}}$ $\qquad x \in (0,2)$

Σ : $\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = \rho^{\frac{1}{2}} \end{cases}$ ⚠



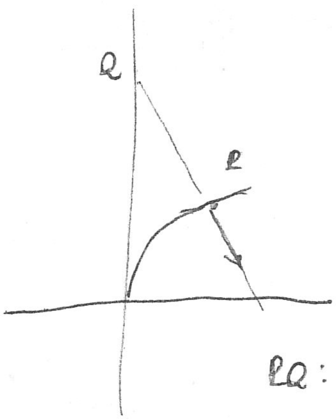
ruotiamo $\mathcal{C}$ attorno all'asse z

$P : (1,0,1)$

Calcoliamo $K(P)$. Utilizziamo il teorema della grannormale

retta PQ:

$z = x^{\frac{1}{2}}$ $\qquad z' = \frac{1}{2} x^{-\frac{1}{2}}$ $\qquad ' = \frac{d}{dx}$



$z'(1) = \frac{1}{2} \equiv m$

$m^{\perp} = -\frac{1}{m} = -2$

PQ: $z - 1 = (-2)(x-1)$

$z - 1 = -2x + 2$

$2x + z - 3 = 0$

Q: $\begin{cases} 2x + z - 3 = 0 \\ x = 0 \end{cases} \Rightarrow z = 3 \qquad Q = (0, \underset{y}{0}, 3)$

$N = \overline{PQ} = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$

grannormale

$\boxed{k_2 = -\frac{1}{\sqrt{5}}}$!

curv. normale del // per P

Curvatura del meridiano in P

$z'' = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-\frac{3}{2}} \qquad z''(1) = -\frac{1}{4}$

$k = \frac{z''}{(1 + z'^2)^{3/2}} = \frac{-\frac{1}{4}}{\left(1 + \frac{1}{4}\right)^{3/2}} = -\frac{1}{4} \frac{1}{\left(\frac{5}{4}\right)^{3/2}} = -\frac{1}{4} \frac{1}{5^{3/2} \frac{1}{4^{3/2}}}$

$= -\frac{1}{4} \frac{1}{5^{3/2} \frac{1}{8}} = \frac{-2}{5^{3/2}}$

$\langle - | - \rangle = -\frac{2}{5^{3/2}}$ $\qquad \boxed{k_1 = +\frac{2}{5^{3/2}}}$

$$K = k_1 \cdot k_2 = -\frac{1}{5^{\frac{1}{2}}} \cdot \left(\frac{+2}{5^{-3/2}}\right) = \frac{-2}{5^2} = -\frac{2}{25}$$

$\frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$

ragionevole...

(2) $m(S)_P$

Lavoriamo in $P:(1,0,1)$

$\varphi = 0$
$\cos\varphi = 1$
$\sin\varphi = 0$.

$$\underline{r} = (\rho\cos\varphi, \rho\sin\varphi, \rho^{\frac{1}{2}}) \qquad \underline{r}:(1,0,1)$$

$$\underline{r}_\rho = (\cos\varphi, \sin\varphi, \tfrac{1}{2}\rho^{-\frac{1}{2}}) \qquad \underline{r}_\rho:(1,0,\tfrac{1}{2})$$

$$\underline{r}_\varphi = (-\rho\sin\varphi, \rho\cos\varphi, 0) \qquad \underline{r}_\varphi:(0,1,0)$$

$$\underline{r}_{\rho\varphi} = \underline{r}_{\varphi\rho} = (-\sin\varphi, \cos\varphi, 0) \qquad \underline{r}_{\rho\varphi}:(0,1,0)$$

$$\underline{r}_{\varphi\varphi} = (-\rho\cos\varphi, -\rho\sin\varphi, 0) \qquad \underline{r}_{\varphi\varphi}:(-1,0,0)$$

$$\underline{r}_{\rho\rho} = (0, 0, -\tfrac{1}{4}\rho^{-\frac{3}{2}}) \qquad \underline{r}_{\rho\rho} = (0,0,-\tfrac{1}{4})$$

$$\begin{array}{c} \\ \underline{r}_\rho \\ \underline{r}_\varphi \end{array} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{vmatrix} = -\frac{1}{2}\underline{i} + \underline{k} = (-\tfrac{1}{2}, 0, 1)$$

$$\Rightarrow \underline{N} = \frac{1}{\sqrt{5}}(+1, 0, -2)$$

! con la nostra scelta 

in P:

$E = \langle \underline{r}_\rho, \underline{r}_\rho \rangle = 1 + \frac{1}{4} = \frac{5}{4}$ $\qquad k_1 = \frac{e}{E} = \frac{1}{2\sqrt{5}}\,\frac{4}{5} = \frac{2}{5^{3/2}}$ ✓

$F = 0$

$G = 1$

$e = \langle \underline{N}, \underline{r}_{\rho\rho} \rangle = -\frac{2}{\sqrt{5}}\left(-\frac{1}{4}\right) = \frac{1}{2\sqrt{5}}$ $\qquad k_2 = \frac{g}{G} = -\frac{1}{\sqrt{5}}$ ✓

$f = 0$

$g = -\frac{1}{\sqrt{5}}$

Weingarten $(F = f = 0)$

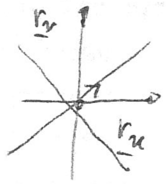
$$\begin{pmatrix} \frac{e}{E} & 0 \\ 0 & \frac{g}{G} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{2}{5^{3/2}} & 0 \\ 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

<u>direzioni principali</u> : indiv. dal meridiano e dal parallelo

meridian. $\langle r_\rho \rangle = \left\langle \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

par. $\langle r_\varphi \rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

<u>direzioni asintotiche</u> $\quad v = \alpha \, r_u + \beta \, r_v$



$$e \alpha^2 + 2f \alpha\beta + g \beta^2 = 0$$

$$\frac{1}{2\sqrt{5}} \alpha^2 - \frac{1}{\sqrt{5}} \beta^2 = 0 \qquad \alpha^2 - 2\beta^2 = 0$$

$$(\alpha \pm \sqrt{2}\beta) = 0$$

$$\alpha = \pm\sqrt{2}\beta$$

<u>dir. asintotiche</u>: $\quad \langle \pm\sqrt{2}\, r_u + r_v \rangle$

$$= \left\langle \begin{pmatrix} \pm 2\sqrt{\frac{2}{5}} \\ \pm\sqrt{\frac{2}{6}} + 1 \\ \pm\sqrt{\frac{2}{6}} \end{pmatrix} \right\rangle$$