GEOMETRIA II a.a. 2016/17

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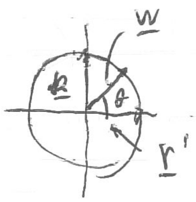
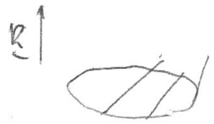
Prova scritta del $\boxed{\text{20 luglio 2017}}$

(1) Sia data la famiglia di superficie rigate Σ_θ ($\theta \in [0, 2\pi]$) di direttrice $\mathcal{C}$: $\underline{r}_{\mathcal{C}} = (\cos s, \sin s, 0)$

e generatrici aventi direzione $\underline{w} = \underline{w}(s) = \cos\theta \, \underline{r}'_{\mathcal{C}}(s) + \sin\theta \cdot \underline{k}$





Dimostrare che Σ_θ è sviluppabile $\iff$ $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi\}$ $\quad ' = \frac{d}{ds}$

(Cosa si ottiene per questi valori? E per gli altri?)

(i) Utilizzare la condizione di sviluppabilità

(ii) Determinare il piano tangente ed esaminarne la variazione lungo le generatrici

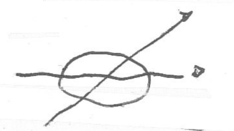
(2) Risolvere l'esercizio (1) attraverso il calcolo della curvatura gaussiana

Tempo a disposizione: 1h 30m

Le risposte vanno adeguatamente giustificate.

① e ② Σ_θ : rigata con direttrice

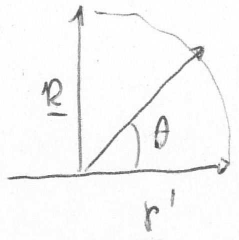
$$\mathscr{C}: \begin{cases} x = \cos s \\ y = \sin s \\ z = 0 \end{cases} \qquad t: \underline{r}_{\mathscr{C}}$$



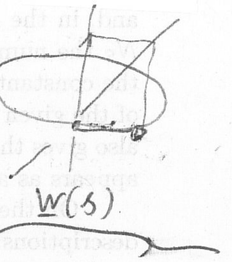
$' = \frac{d}{ds}$

generatrice: direzione $\cos\theta\, \underline{r}' + \sin\theta\, \underline{k}$

$$\underline{r}'_{\mathscr{C}} = (-\sin s, \cos s, 0)$$



$\theta \in [0, 2\pi]$



$\underline{w}(s)$

$$\Sigma_\theta : \quad \underline{r}(s,t) = \underline{r}_{\mathscr{C}}(s) + t\left[\cos\theta \cdot \underline{r}'_{\mathscr{C}}(s) + \sin\theta \cdot \underline{k}\right]$$

Dimostriamo che Σ_θ è sviluppabile $\iff \theta = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi$

Che cosa si ottiene per questi valori e per gli altri?

$$\underline{r}(s,t) = \Big(\underbrace{\cos s + t\cos\theta(-\sin s)}_{x},\ \sin s + t\cos\theta \cos s,\ \underbrace{t\sin\theta}_{z}\Big)$$

$$= \left(\cos s - t\cos\theta\sin s,\ \sin s + t\cos\theta\cos s,\ t\sin\theta\right)$$

verifichiamola in vari modi

$$\underline{w}_s = \begin{pmatrix} -\cos\theta\cos s \\ -\cos\theta\sin s \\ 0 \end{pmatrix}$$

(i) tramite $\left|\underline{r}'_{\mathscr{C}}, \underline{w}', \underline{w}'_s\right| = 0$

$$\begin{vmatrix} -\sin s & \cos\theta(-\sin s) & -\cos\theta\cos s \\ \cos s & \cos\theta \cdot \cos s & -\cos\theta\sin s \\ 0 & \sin\theta & 0 \end{vmatrix} = 0$$

$$\sin\theta \begin{vmatrix} -\sin s & -\cos\theta\cos s \\ \cos s & -\cos\theta\sin s \end{vmatrix} = 0$$

$$\sin\theta \cdot \left[\cos\theta \sin^2 s + \cos\theta \cos^2 s \right] = 0$$

$$\sin\theta \cos\theta = 0 \qquad (\Leftrightarrow \sin 2\theta = 0)$$

$$\Rightarrow \qquad \theta = 0, \frac{\pi}{2}, \pi, \frac{3}{2\pi} \quad (2\pi)$$

(ii) Determiniamo il piano tangente in un generico pto

$$\underline{r}_s = (-\sin s - t\cos\theta\cos s, \ \cos s - t\cos\theta \sin s, \ 0)$$

$$\underline{r}_t = (-\cos\theta \sin s, \ \cos\theta\cos s, \ \sin\theta) \qquad \|\underline{r}_t\| = 1$$

$$\underline{r}_s \times \underline{r}_t = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin s - t\cos\theta\cos s & \cos s - t\cos\theta\sin s & 0 \\ -\cos\theta \sin s & +\cos\theta\cos s, & \sin\theta \end{vmatrix}$$

$$= \underline{i}\left[\cos s - t\cos\theta\sin s\right]\sin\theta - \underline{j}\left[-\sin s - t\cos\theta\cos s\right]\sin\theta$$

$$+ \underline{k}\left[\underbrace{\cancel{-\sin s\cos s}\cos\theta - t\cos^2\theta\cos^2 s + \cancel{\sin s\cos s}\cos\theta - t\cos^2\theta\sin^2 s}_{-t\cos^2\theta}\right]$$

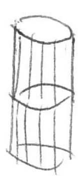
$\sin\theta = 0$ Sup. piana

$\underline{N}$ costante $z = 0$

eccettuato l'interno del cerchio

$\cos\theta = 0$ $\underline{N}$ ind. da t.

$\underline{N}$ costante lungo le generatrici



(iii) Calcoliamo K (e imponiamola $= 0$)

$$\underline{r}_{ss} = (-\cos s + t\cos\theta \sin s,\ -\sin s - t\cos\theta \cos s,\ 0)$$

$$\underline{r}_{st} = (-\cos\theta\cos s,\ -\cos\theta \sin s,\ 0)$$

$$\underline{r}_{tt} = 0$$

$$g = \langle \underline{r}_{tt}, \underline{N} \rangle = 0$$

poniamo

$$K = \frac{eg - f^2}{EG - F^2}$$

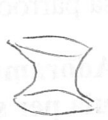
$$\begin{array}{c} \underline{r}_t \\ \underline{r}_s \\ \underline{r}_{st} \end{array} \begin{vmatrix} -\cos\theta\sin s & \cos\theta\cos s & \sin\theta \\ -\sin s - t\cos\theta\cos s & \cos s - t\cos\theta\sin s & 0 \\ -\cos\theta\cos s & -\cos\theta\sin s & 0 \end{vmatrix} = 0$$

$$\sin\theta \left\{ (+\cos\theta)\sin^2 s + \cancel{t\cos^2\theta\cos s\sin s} + \cos\theta\cos^2 s - \cancel{t\cos^2\theta\cos s\sin s} \right\}$$

$\Rightarrow \sin\theta\cos\theta = 0$ e si ritrova il risultato precedente.

Tipi di superficie

Σ_θ : iperboloide rigato $\quad \theta \notin \{0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi\}$



$\Sigma_0 = \Sigma_{2\pi}$ = piano

Surp. tangente

$\Sigma_{\frac{\pi}{2}} = \Sigma_{\frac{3}{2}\pi}$ = cilindro circ. retto

$\cos\theta = 0 \quad x^2+y^2=1$

$\sin\theta = 0 \quad z=0$

altrimenti: iperboloide iperbolico (rigato)

$$x^2 = \cos^2 s + t^2\cos^2\theta\sin^2 s - 2t\cos s\sin s\cos\theta$$

$$y^2 = \sin^2 s + t^2\cos^2 s\cos^2\theta + 2t\cos\theta\cos s\sin s$$

$$x^2+y^2 = 1 + t^2\cos^2\theta \qquad t^2 = \frac{1}{\cos^2\theta}(x^2+y^2-1) \qquad \cos\theta \neq 0$$

$$z^2 = t^2\sin^2\theta \qquad z^2 = \frac{\sin^2\theta}{\cos^2\theta}(x^2+y^2-1)$$

$$\boxed{z^2 - \tan^2\theta\,(x^2+y^2-1) = 0}$$