

Prova scritta del 16 gennaio 2019

① Nel piano, si calcoli  $L_X \omega$  in due modi,  
 con  $\omega = dx \wedge dy$ ,  $X = \sin x \frac{\partial}{\partial x} + \sin y \frac{\partial}{\partial y}$

② In  $(0, 2\pi) \times (0, \pi)$ , dati  $\omega = \sin \theta d\theta \wedge d\varphi$ ,  
 $X = f(\theta) \frac{\partial}{\partial \theta}$  ( $f = f(\theta)$  liscia),

si determini  $f$  in modo che  $L_X \omega = 0$  [tutti]

Dire se  $X$  così determinato risulta  
 hamiltoniano (rispetto a  $\omega$ )

③ In  $(0, 2\pi) \times (0, \pi)$ , data  
 $g = \sin^2 \theta d\varphi^2 + d\theta^2$   
 e dato  $X = f(\varphi) \frac{\partial}{\partial \varphi}$  ( $f = f(\varphi)$  liscia),

per quali  $f$   $X$  risulta essere di Killing  
 per  $g$ ?

Tempo a disposizione: 1h

Le risposte vanno adeguatamente giustificate

$$\textcircled{1} \quad X = \overset{x_1}{\sin \alpha} \frac{\partial}{\partial x} + \overset{x_2}{\sin y} \frac{\partial}{\partial y}$$

$$\omega = dx \wedge dy$$

calcolare  $L_X \omega$  in due modi

$$\begin{aligned} \textcircled{1} \quad \text{diretto} \quad L_{X_1} dx \wedge dy &= L_{X_1} dx \wedge dy + dx \wedge L_{X_1} dy \\ &= d L_{X_1}(\alpha) \wedge dy + dx \wedge d L_{X_1}(y) \\ &= d \sin(\alpha) \wedge dy + dx \wedge d(\sin(y)) \\ &= d \overset{1}{\sin \alpha} \frac{\partial}{\partial \alpha} \wedge dy \quad \underbrace{\hspace{10em}}_{=0} \\ &= \cos \alpha \, dx \wedge dy \end{aligned}$$

$$\begin{aligned} L_{X_2}(dx \wedge dy) &= L_{X_2} dx \wedge dy + dx \wedge L_{X_2} dy \\ &= d L_{X_2} \alpha \wedge dy + dx \wedge d L_{X_2}(y) \\ &= dx \wedge d \sin y \\ &= \cos y \, dx \wedge dy \end{aligned}$$

$$\begin{aligned} L_X \omega &= (\cos \alpha + \cos y) \, dx \wedge dy \\ &= (\cos \alpha + \cos y) \, \omega \end{aligned}$$

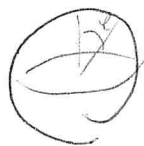
$$\textcircled{2} \quad i_X \omega = i_{X_1} \omega + i_{X_2} \omega = \sin \alpha \, dy - \sin y \, dx$$

$$\begin{aligned} d\omega = 0 \quad d i_X \omega &= \cos \alpha \, dx \wedge dy - \cos y \, dy \wedge dx \\ i_X d\omega = 0 &= (\cos \alpha + \cos y) \, dx \wedge dy \quad \wedge \end{aligned}$$

②

$$\omega = \sin \theta \, d\theta \wedge d\varphi$$

$$X = f(\theta) \frac{\partial}{\partial \theta}$$



Determinare  $f$  in modo che  $L_X \omega = 0$

Cio' equivale a  $d(i_X \omega) = 0$  ( $d\omega = 0$ )

$$i_X \omega = f \sin \theta \, d\varphi$$

$$d(f \sin \theta \, d\varphi) = 0$$

$$\Rightarrow d(f(\theta) \sin \theta) \wedge d\varphi = 0$$

$$\stackrel{1 = \frac{d}{d\theta}}{\Rightarrow} (f' \sin \theta + f \cos \theta) d\theta \wedge d\varphi = 0$$

$$\Rightarrow f' \sin \theta + f \cos \theta = 0$$

$$\frac{f'}{f} = - \frac{\cos \theta}{\sin \theta} \quad (= -\cot \theta) \quad \theta \in (0, \pi)$$

$$d \log f = - \frac{\cos \theta}{\sin \theta} d\theta = - d(\log \sin \theta)$$

$$\log f = -\log \sin \theta + c = \log (\sin \theta)^{-1} + c$$

$$f = c \frac{1}{\sin \theta} \quad c > 0$$

$X$  è hamiltoniana su  $(S^2, \omega)$

$$i_X \omega = c \, d\varphi \quad \leadsto \mathcal{H} = c\varphi + \cos \theta$$

Hamiltoniana

$$\textcircled{3} \quad g = \sin^2 \theta d\varphi^2 + d\theta^2$$

$$X = f(\varphi) \frac{\partial}{\partial \varphi} \quad \text{Ea quali } f \text{ } X \text{ \u00e9 Killing?}$$

$$\left[ \begin{aligned} \mathcal{L}_X g &= \mathcal{L}_X (\sin^2 \theta d\varphi^2) + \underbrace{\mathcal{L}_X d\theta^2}_{=0} \\ &= \end{aligned} \right.$$

$$\underbrace{\mathcal{L}_X (\sin^2 \theta)}_{=0} d\varphi^2 + \sin^2 \theta \mathcal{L}_X d\varphi^2$$

$$= \sin^2 \theta (\mathcal{L}_X d\varphi \cdot d\varphi + d\varphi \cdot \mathcal{L}_X d\varphi)$$

$$= \sin^2 \theta (dX(\varphi) d\varphi + d\varphi dX(\varphi))$$

$$= \sin^2 \theta (df d\varphi + d\varphi df)$$

$$= \sin^2 \theta \cdot 2 \cdot f' d\varphi^2$$

$$= 2f' \sin^2 \theta d\varphi^2$$

$$\mathcal{L}_X g = 0 \iff f' = 0 \implies f = \text{const}$$