ISTITUZIONI DI GEOMETRIA SUPERIORE

a.a. 2018/19 Prof. M. Spera

Prova scritta del 16 gennaio 2019

① Nel piano, si calcoli $\mathcal{L}_X \omega$ in due modi, con $\omega = dx \wedge dy$, $X = \sin x \frac{\partial}{\partial x} + \sin y \frac{\partial}{\partial y}$

② In $(0, 2\pi) \times (0, \pi)$, dati $\omega = \sin\theta \, d\theta \wedge d\varphi$,

$X = f(\theta) \frac{\partial}{\partial \theta}$ ($f = f(\theta)$ liscia),

si determini f in modo che $\mathcal{L}_X \omega = 0$ [tutti]

Dire se X così determinato risulta hamiltoniano (rispetto a ω)

③ In $(0, 2\pi) \times (0, \pi)$, data

$$g = \sin^2\theta \, d\varphi^2 + d\theta^2$$

e dato $X = f(\varphi) \frac{\partial}{\partial \varphi}$ ($f = f(\varphi)$ liscia),

per quali f X risulta essere di Killing per g?

Tempo a disposizione: 1h

Le risposte vanno adeguatamente giustificate

(1) $X = \underbrace{\sin x \frac{\partial}{\partial x}}_{X_1} + \underbrace{\sin y \frac{\partial}{\partial y}}_{X_2}$

$\omega = dx \wedge dy$

Calcolare $\mathcal{L}_X \omega$ in due modi

(1) diretto

$$\mathcal{L}_{X_1} dx \wedge dy = \mathcal{L}_{X_1} dx \wedge dy + dx \wedge \mathcal{L}_{X_1} dy$$

$$= d\,\mathcal{L}_{X_1}(x) \wedge dy + dx \wedge d\,\mathcal{L}_{X_1}(y)$$

$$= d\, X_1(x) \wedge dy + dx \wedge \underbrace{d(\underbrace{X_1(y)}_{=0})}_{=0}$$

$$= d\, \sin x \overbrace{\frac{\partial x}{\partial x}}^{1} \wedge dy$$

$$= \cos x\, dx \wedge dy$$

$$\mathcal{L}_{X_2}(dx \wedge dy) = \mathcal{L}_{X_2} dx \wedge dy + dx \wedge \mathcal{L}_{X_2} dy$$

$$= \underbrace{d\,\mathcal{L}_{X_2} x}_{=0} \wedge dy + dx \wedge d\,\mathcal{L}_{X_2}(y)$$

$$= dx \wedge d\, X_2(y) = dx \wedge d \sin y$$

$$= \cos y\, dx \wedge dy$$

$$\mathcal{L}_X \omega = (\cos x + \cos y)\, dx \wedge dy$$

$$= (\cos x + \cos y)\, \omega$$

(2) $i_X \omega = i_{X_1} \omega + i_{X_2} \omega = \sin x\, dy - \sin y\, dx$

$d\omega = 0$

$i_X d\omega = 0$

$d\, i_X \omega = \cos x\, dx \wedge dy - \cos y\, dy \wedge dx$

$= (\cos x + \cos y)\, dx \wedge dy$

(2)

$\omega = \sin\theta \, d\theta \wedge d\varphi$

$X = f(\theta)\frac{\partial}{\partial\theta}$

Determinare f in modo che $\mathcal{L}_X \omega = 0$

Ciò equivale a $\quad d(i_X\omega) = 0 \qquad (d\omega = 0)$

$i_X\omega = f \sin\theta \, d\varphi$

$d(f\sin\theta \, d\varphi) = 0$

$\Rightarrow \quad d(f(\theta)\sin\theta) \wedge d\varphi = 0$

$' = \frac{d}{d\theta} \quad (f'\sin\theta + f\cos\theta)\, d\theta \wedge d\varphi = 0$

$\Rightarrow \quad f'\sin\theta + f\cos\theta = 0$

$$\frac{f'}{f} = -\frac{\cos\theta}{\sin\theta} \quad (= -\cot\theta) \quad \theta \in (0,\pi)$$

$$d\log f = -\frac{\cos\theta \, d\theta}{\sin\theta} = -d(\log\sin\theta)$$

$$\log f = -\log\sin\theta + c = \log((\sin\theta)^{-1}) + c$$

$$f = c\,\frac{1}{\sin\theta} \qquad c > 0$$

X è hamiltoniano su (S^2, ω)

$i_X\omega = c\, d\varphi \quad \rightsquigarrow \lambda = c\varphi + \text{cost}$

Hamiltoniana

(3) $g = \sin^2\theta \, d\varphi^2 + d\theta^2$

$X = f(\varphi)\frac{\partial}{\partial\varphi}$ per quali f X è Killing?

$$\mathcal{L}_X g = \mathcal{L}_X(\sin^2\theta \, d\varphi^2) + \underbrace{\mathcal{L}_X d\theta^2}_{= \dots = 0}$$

$$= \underbrace{\mathcal{L}_X(\sin^2\theta)}_{=0} d\varphi^2 + \sin^2\theta \, \mathcal{L}_X d\varphi^2$$

$$= \sin^2\theta \, (\mathcal{L}_X d\varphi \cdot d\varphi + d\varphi \cdot \mathcal{L}_X d\varphi)$$

$$= \sin^2\theta \, (d\, X(\varphi) \, d\varphi + d\varphi \, d\, X(\varphi))$$

$$= \sin^2\theta \, (df \, d\varphi + d\varphi \, df)$$

$$= \sin^2\theta \cdot 2 \cdot f' \, d\varphi^2$$

$$= 2f' \sin^2\theta \, d\varphi^2$$

$$\mathcal{L}_X g = 0 \iff f' = 0 \implies f = \text{cost}$$