ISTITUZIONI DI GEOMETRIA SUPERIORE

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(1) In $\mathbb{R}^3$, siano dati $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

e $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$

Calcolare $\mathcal{L}_X \omega$ con la definizione

(2) Calcolare $\mathcal{L}_X \omega$ con la formula di Cartan

(3) In $\mathbb{R}^3$ sia data la distribuzione Δ individuata da $\omega = x\, dy + y\, dz + z\, dx = 0$.

Dire se Δ è integrabile. Determinarne in seguito una base locale, verificandone il carattere involutorio o meno.

Tempo a disposizione: 1 h

Le risposte vanno adeguatamente giustificate

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(1) In $\mathbb{R}^3$ siano

(2) $$\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$$

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

Calcolare $\mathcal{L}_X \omega$ in due modi diversi

(1) diretto $\mathcal{L}_{X_1}\omega = \mathcal{L}_{\frac{\partial}{\partial x}}(x\,dy\wedge dz) + \mathcal{L}_{\frac{\partial}{\partial x}}(y\,dz\wedge dx) + \mathcal{L}_{\frac{\partial}{\partial x}}(z\,dx\wedge dy)$ $\quad X_1 = \frac{\partial}{\partial x}$

$$\mathcal{L}_{\frac{\partial}{\partial x}}(x\,dy\wedge dz) = \underbrace{\mathcal{L}_{\frac{\partial}{\partial x}}(x)}_{\frac{\partial x}{\partial x}=1} dy\wedge dz + x\underbrace{\mathcal{L}_{\frac{\partial}{\partial x}} dy}_{d\frac{\partial y}{\partial x}=0}\wedge dz + x\,dy\wedge \underbrace{\mathcal{L}_{\frac{\partial}{\partial x}} dz}_{0}$$

$$= dy\wedge dz$$

$$\mathcal{L}_{\frac{\partial}{\partial x}}(y\,dz\wedge dx) = \underbrace{\mathcal{L}_{\frac{\partial}{\partial x}} y\, dz\wedge dx}_{0} + y\underbrace{\mathcal{L}_{\frac{\partial}{\partial x}} dz}_{d\mathcal{L}_{\frac{\partial}{\partial x}}z=0}\wedge dx + y\,dz\wedge \underbrace{\mathcal{L}_{\frac{\partial}{\partial x}} dx}_{d1=0}$$

$$\mathcal{L}_{\frac{\partial}{\partial x}}(z\,dx\wedge dy) = \dots = 0$$

$\mathcal{L}_{X_1}\omega = dy\wedge dz$ Similmente $\mathcal{L}_{X_2}\omega = dz\wedge dx$, $X_2 = \frac{\partial}{\partial y}$

$\mathcal{L}_{X_3}\omega = dx\wedge dy$, $X_3 = \frac{\partial}{\partial z}$

$\Rightarrow$ $\mathcal{L}_X\omega = dx\wedge dy + dy\wedge dz + dz\wedge dx$

② Cartan

$$d\omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy$$
$$= 3\, dx \wedge dy \wedge dz$$

Ⓐ $$i_X d\omega = 3\, dy \wedge dz \overset{+\,3\, dz \wedge dx}{\cancel{-\,3\, dx \wedge dz}} + 3\, dx \wedge dy$$

$$i_{X_1} \omega = 0 - y\, dz + z\, dy$$

$$d\, i_{X_1} \omega = -dy \wedge dz + dz \wedge dy$$
$$= -2\, dy \wedge dz$$

$$d\, i_{X_2} \omega = d(x\, dz - z\, dx)$$
$$= dx \wedge dz - dz \wedge dx = -2\, dz \wedge dx$$

$$d\, i_{X_3} \omega = -2\, dx \wedge dy$$

Ⓑ $$d\, i_X \omega = -2\,(dy \wedge dz + dz \wedge dx + dx \wedge dy)$$

$$Ⓐ + Ⓑ = dx \wedge dy + dy \wedge dz + dz \wedge dx \quad \checkmark$$

③ In $\mathbb{R}^3$ sia data la distribuzione

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$$\omega = x\,dy + y\,dz + z\,dx$$

È integrabile? $\qquad d\omega = dx \wedge dy + dy \wedge dz + dz \wedge dx$

$$\omega \wedge d\omega = y\,\underbrace{dz \wedge dx \wedge dy}_{= dx \wedge dy \wedge dz} + z\,dx \wedge dy \wedge dz + x\,\underbrace{dy \wedge dz \wedge dx}_{= dx \wedge dy \wedge dz} = (x+y+z)\,dx \wedge dy \wedge dz$$

$\Rightarrow$ NO

Determinare una base locale per la distribuzione verificando il suo carattere non involutorio

$$X = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + \gamma \frac{\partial}{\partial z} \qquad \omega(X) = 0$$

$$\overset{1\,2}{x\beta} + \overset{3\,1}{z\alpha} + \overset{3\,\,2}{\gamma y} = 0$$

sia $\gamma = 0 \qquad \alpha = x \qquad \beta = -z \qquad X_1 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}$

$ \beta = 0 \qquad \alpha = y \qquad \gamma = -z \qquad X_2 = y\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}$

$$[X_1, X_2] = -y\frac{\partial}{\partial x} - z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} = -(y+z)\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$$

$$\begin{vmatrix} x & -z & 0 \\ y & 0 & -z \\ -(y+z) & 0 & z \end{vmatrix} = -\begin{vmatrix} -z & x & 0 \\ 0 & y & -z \\ 0 & -(y+z) & z \end{vmatrix} = +z\begin{vmatrix} y & -z \\ -(y+z) & z \end{vmatrix}$$

$$= z\,(yz - (y+z)z)$$
$$= z(-z^2) = -z^3 \neq 0$$