ISTITUZIONI DI GEOMETRIA SUPERIORE II

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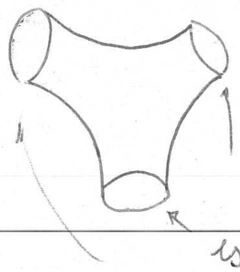
① Sia $X =$



Determinare $H^*(X)$

esclusi i bordi, e "vuoto"

② Siano, in $\mathbb{R}^3$, $\omega = x^3\, dy \wedge dz + y^3\, dz \wedge dx + z^3\, dx \wedge dy$

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

calcolare $\mathcal{L}_X \omega$ in due modi diversi

③ In $\mathbb{R}^3$, dimostrare che

$$\omega = dy \wedge dz + dz \wedge dx + dx \wedge dy$$

è esatta, e determinare $\chi = \alpha\, dx + \beta\, dy + \gamma\, dz$ tale che $\omega = d\chi$. χ è unica?

Spiegare.

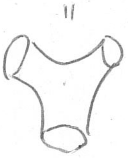
Tempo a disposizione: 1h.

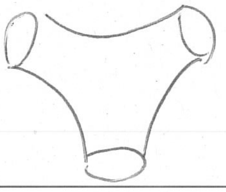
Le risposte vanno adeguatamente giustificate.

Istigeo II

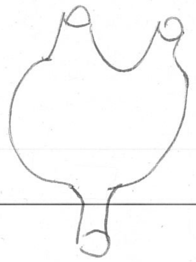
30/1/14

X
$\parallel$

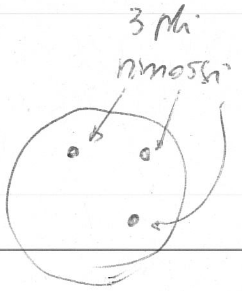
① $H^*($  $)$ "trinione"
"trinion"



$\approx$



$\approx$



$\approx \quad \mathbb{R}^2 \setminus \{p,q\}$

due pti
rimossi

ma $H^*(\mathbb{R}^2 \setminus \{p,q\})$ è nota :

$H^0 = \mathbb{R}$ connesso
$H^1 = \mathbb{R}^2$
$H^2 = 0$ non compatto

② $$w = \overbrace{x^3 \, dy \wedge dz}^{①} + \overbrace{y^3 dz \wedge dx}^{②} + \overbrace{z^3 dx \wedge dy}^{③}$$

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$$\mathcal{L}_X ① = \mathcal{L}_X(x^3)\, dy \wedge dz + x^3 \mathcal{L}_X dy \wedge dz + x^3 \, dy \wedge \mathcal{L}_X dz$$

$$= \underset{\underset{3x^2}{\overset{\parallel}{\frac{\partial x^3}{\partial x}}}}{X(x^3)}\, dy \wedge dz + x^3 \, d \underset{\underset{= d(1) = 0}{d\left(\frac{\partial y}{\partial y}\right)}}{\underbrace{\mathcal{L}_X y}} \wedge dz + x^3 dy \wedge d \underset{\underset{0}{\parallel}}{\mathcal{L}_X z}$$

$\mathcal{L}_X ① = 3x^2 \, dy \wedge dz$ analogamente:
$\mathcal{L}_X ② = 3y^2 \, dz \wedge dx$ $\Rightarrow \mathcal{L}_X w = 3x^2 dy \wedge dz$
$\mathcal{L}_X ③ = 3z^2 \, dx \wedge dy$ + perm. cicliche

Con Cartan:

$\underline{i_X \omega|}$ $\quad i_X$ ① $= i_X (x^3 dy \wedge dz) =$

$$= x^3 \, i_X (dy \wedge dz) = x^3 \{ \underset{\substack{\| \\ dy(X) \\ = X(y) \\ \| \\ 1}}{(i_X dy)} \wedge dz - \underset{\substack{\| \\ 1}}{(i_X dz)} \wedge dy \}$$

$$= x^3 (dz - dy)$$

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i_X ① $= x^3 (dz - dy)$

i_X ② $= y^3 (dx - dz)$

i_X ③ $= z^3 (dy - dx)$

$\underline{d(i_X \omega)|}$ $= d(x^3(dz - dy)) +$ termini analoghi

$$= (d x^3) \wedge (dz - dy) + x^3 \underbrace{d(dz - dy)}_{\substack{d^2z - d^2y \\ \| \\ 0}}$$

$$= 3x^2 \, dx \wedge (dz - dy) \quad + \dots$$

$$= 3x^2 \, dx \wedge dz - 3x^2 \, dx \wedge dy \quad + \dots$$

$$3y^2 \, dy \wedge dx - 3y^2 \, dy \wedge dz$$

$$+ \; 3z^2 \, dz \wedge dy - 3z^2 \, dz \wedge dx$$

$\underline{d\omega|}$ $\quad d$ ① $= d(x^3) \wedge dy \wedge dz = 3x^2 \, dx \wedge dy \wedge dz$

$$d\omega = 3(x^2 + y^2 + z^2) \, dx \wedge dy \wedge dz$$

$$\underline{i_X d\omega|} = 3(x^2 + y^2 + z^2) \{ \underset{\substack{\| \\ dx(X) \\ = X(x) = 1}}{\underbrace{i_X}} dx \wedge dy \wedge dz - i_X dy \wedge dx \wedge dz + i_X dz \wedge dx \wedge dy \}$$

$$= 3(x^2 + y^2 + z^2) \{ dy \wedge dz + dz \wedge dx + dx \wedge dy \}$$

$\boxed{(di_X + i_X d)w}$ coeff di $dx \wedge dy$

$$-3x^2 - 3y^2 + 3x^2 + 3y^2 + 3z^2 = 3z^2$$

e così gli altri. ✓

③ $w = dy \wedge dz + dz \wedge dx + dx \wedge dy$

w è banalmente chiusa => è esatta.

$$w = d\,(\overbrace{\alpha\, dx + \beta\, dy + \gamma\, dz}^{\chi})$$

$$= d\alpha \wedge dx + d\beta \wedge dy + d\gamma \wedge dz$$

$$= \left(\frac{\partial \alpha}{\partial y} dy + \frac{\partial \alpha}{\partial z} dz\right) \wedge dx + \left(\frac{\partial \beta}{\partial x} dx + \frac{\partial \beta}{\partial z} dz\right) \wedge dy + \left(\frac{\partial \gamma}{\partial x} dx + \frac{\partial \gamma}{\partial y} dy\right) \wedge dz$$

$$= \left(\frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}\right) dz \wedge dx + \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dx \wedge dy$$

$$+ \left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}\right) dy \wedge dz$$

$$\Rightarrow \begin{cases} \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} = 1 \\ \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} = 1 \\ \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} = 1 \end{cases}$$

=> possiamo scegliere

$\alpha = z - x$

$\beta = x - y$

$\gamma = y - z$

$$\chi = (z-x)dx + (x-y)dy + (y-z)dz$$

tale scelta non è unica!

$\chi \mapsto \chi + df$ ↖ arbitraria dà la stessa sol.