ISTITUZIONI DI GEOMETRIA SUPERIORE II

Prof. Mauro Spera - UCSC, Brescia

Prova scritta del 23 luglio, 2014

① Dato il tensore $T = \sin\varphi \frac{\partial}{\partial r} \otimes d\vartheta \otimes e^{r} d\varphi$,

e il campo vettoriale $X = e^{\varphi} \frac{\partial}{\partial r}$, $\quad \vartheta, \varphi, r \in \mathbb{R}$

Calcolare $\mathcal{L}_X T$

② Data la varietà simplettica $(\mathbb{R}^2, \omega = dq \wedge dp)$ e $H = q^6 + p^6$, determinare X_H (gradiente simplettico di H). Verificare direttamente che $\mathcal{L}_{X_H} \omega = 0$. Si individuino le curve integrali di X_H e si provi che, rispetto alla metrica standard su $\mathbb{R}^2$, ∇H (gradiente riemanniano) è ortogonale a X_H punto per punto.

③ In $\mathbb{R}^3$, dato $w = y^2 dx + f(y) dz$, $\quad y > 0$, $f > 0$

determinare f in modo che la distribuzione definita da $w = 0$ risulti integrabile.

Per una f fissata, individuare le sottovarietà integrali della distribuzione e una sua base locale [Si verifichi, a posteriori, che la distribuzione è involutiva]

Tempo a disposizione: 1h

Le risposte vanno adeguatamente giustificate.

1 $(\varphi, \vartheta, r) \in \mathbb{R}^3$

$$T = \sin\varphi \frac{\partial}{\partial r} \otimes d\vartheta \otimes e^r d\varphi \qquad \left(= \sin\varphi\, e^r \frac{\partial}{\partial r} \otimes d\vartheta \otimes d\varphi\right)$$

$$X = e^\varphi \frac{\partial}{\partial r}$$

$\boxed{\text{Compute } \mathcal{L}_X T}$ answer $\left(= e^{\varphi + r} \sin\varphi \frac{\partial}{\partial r} \otimes d\vartheta \otimes d\varphi\right)$

$$\begin{aligned}
\mathcal{L}_X T = \; & \mathcal{L}_X(\sin\varphi\, e^r) \frac{\partial}{\partial r} \otimes d\vartheta \otimes d\varphi && (1) \\
&+ (\sin\varphi\, e^r)\, \mathcal{L}_X\left(\frac{\partial}{\partial r}\right) \otimes d\vartheta \otimes d\varphi && (2) \\
&+ (\sin\varphi\, e^r) \frac{\partial}{\partial r} \otimes \mathcal{L}_X(d\vartheta) \otimes d\varphi && (3) \\
&+ (\sin\varphi\, e^r) \frac{\partial}{\partial r} \otimes (d\vartheta) \otimes \mathcal{L}_X d\varphi && (4)
\end{aligned}$$

(1): $\mathcal{L}_X(\sin\varphi\, e^r) = X(\sin\varphi\, e^r) = e^\varphi \frac{\partial}{\partial r}(\sin\varphi\, e^r)$

$= e^\varphi \sin\varphi \frac{\partial}{\partial r} e^r = e^\varphi e^r \sin\varphi \quad (= e^{\varphi + r} \sin\varphi)$

$\Rightarrow \boxed{(1) = e^{\varphi + r} \sin\varphi \frac{\partial}{\partial r} \otimes d\vartheta \otimes d\varphi}$

(2): $\mathcal{L}_X \frac{\partial}{\partial r} = \left[X, \frac{\partial}{\partial r}\right] = \left[e^\varphi \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right] = \ldots = 0$

$\boxed{(2) = 0}$

(3): $\mathcal{L}_X d\vartheta = d\mathcal{L}_X \vartheta = d\, X(\vartheta) = d\left(e^\varphi \frac{\partial \vartheta}{\partial r}\right) = 0$

$\Rightarrow (3) = 0$

(4): $\mathcal{L}_X d\varphi = d\mathcal{L}_X \varphi = d\, X(\varphi) = d\left(e^\varphi \frac{\partial \varphi}{\partial r}\right) = 0 \Rightarrow \boxed{(4) = 0}$

$\Rightarrow \mathcal{L}_X T = (1)$

(2)

$(\mathbb{R}^2, \omega = dq \wedge dp) \qquad H = q^6 + p^6$

find X_H (Hamiltonian v. f. associated with H)

$$dH = 6q^5\, dq + 6p^5\, dp \qquad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Omega^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

find X_H: $\quad i_{X_H}\omega = dH$

$$\Omega^{-T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (= \Omega)$$

$X_H = \Omega^{-T}\nabla H \leftarrow$ abuse of notation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 6q^5 \\ 6p^5 \end{pmatrix} = \begin{pmatrix} 6p^5 \\ -6q^5 \end{pmatrix}$$

$$\Rightarrow \boxed{X_H = 6p^5 \frac{\partial}{\partial q} - 6q^5 \frac{\partial}{\partial p}}$$

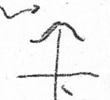
integral curves of X_H: $H = c$

$\rightsquigarrow$ via implicit calculus

$q^6 + p^6 = c$

$p = p(q)$ if $\frac{\partial H}{\partial p} = 6p^5 \neq 0$

$p' = -\frac{q^5}{p^5} \rightsquigarrow$



Verify directly that $\mathcal{L}_{X_H}\omega = 0$

$$\mathcal{L}_{X_H}(dq \wedge dp) = \mathcal{L}_{X_H}(dq) \wedge dp + dq \wedge \mathcal{L}_{X_H}(dp)$$

$$= d\,\mathcal{L}_{X_H}(q) \wedge dp + dq \wedge d\,\mathcal{L}_{X_H}(p)$$

$$= d\,X_H(q) \wedge dp + dq \wedge d\,X_H(p)$$

Now $\quad d\,X_H(q) = d(6p^5 \overset{=1}{\frac{\partial q}{\partial q}})$

$$= 6 \cdot 5\, p^4\, dp = 30p^4\, dp$$

$\Rightarrow d\,X_H(q) \wedge dp = 0$. Similarly $dq \wedge d\,X_H(p) = 0$

$\Rightarrow \mathcal{L}_{X_H}\omega = 0$. Finally, if $g = dx^2 + dy^2$

$$\boxed{\nabla H = 6q^5 \frac{\partial}{\partial q} + 6p^5 \frac{\partial}{\partial p}}$$

and $\langle \nabla H, X_H \rangle = .. = 0$

③

$$w = y^2 dx + f(y) dz \qquad y > 0, \quad f > 0$$

Integrability condition: $w \wedge dw = 0$

$$' = \frac{d}{dy}$$

$$dw = 2y\, dy \wedge dx + f'(y)\, dy \wedge dz$$

$$w \wedge dw = 2yf\, dz \wedge dy \wedge dx + y^2 f'\, dx \wedge dy \wedge dz$$

$$= (-2yf + y^2 f')\, dx \wedge dy \wedge dz = 0$$

$$\Rightarrow \quad -2f + yf' = 0$$

$$\frac{f'}{f} = \frac{2}{y} \qquad \frac{df}{f} = \frac{2}{y} dy$$

$$\Rightarrow \log|f| = \log f = 2\log|y| + c = 2\log y + c$$
$$= \log y^2 + c$$

$$\Rightarrow f = C y^2 \qquad C = e^c > 0$$

Choose $C = 1$, $\quad \cancel{y^2}(dx + dz) = 0 \qquad dx + dz = 0$

$$d(x+z) = 0 \qquad x + z = c \quad \leftarrow \text{integral submanifolds (planes)}$$

$w(X) = 0$

local basis:

$$(dx + dz,\ \overbrace{\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + \gamma \frac{\partial}{\partial z}}^{X}) = 0$$

$$\alpha + \gamma = 0 \qquad \gamma = -\alpha \qquad X = \alpha \overbrace{\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right)}^{X_1} + \beta \underbrace{\frac{\partial}{\partial y}}_{X_2}$$

(X_1, X_2) local basis.

Clearly $[X_1, X_2] = 0$ (involutive, as expected)