TOPOLOGIA E GEOMETRIA DIFFERENZIALE

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① Siano dati, su $\mathbb{R}^2$

$$g = dx^2 + dy^2 \quad e \quad X = x\frac{\partial}{\partial y} - y^{\alpha}\frac{\partial}{\partial x}$$

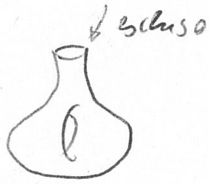
Determinare $\alpha \in \mathbb{R}$ in modo che X sia di Killing

② In $M = \mathbb{R}^2 - \{(0,0)\}$ sia data

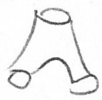
$$\omega = 2\,\frac{-y}{x^2+y^2}\,dx + \left(\frac{2x}{x^2+y^2} + y^2 + 1\right)dy$$

Si dica se $\omega \in Z^1(M)$, $\omega \in B^1(M)$.

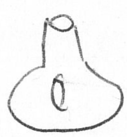
Si determini $[\omega] \in H^1 = Z^1/B^1$

③ Determinare $H^*(M)$, $M =$ 

Sugg. Si usi Mayer - Vietoris

1. $M =$  $U =$  $V =$ 

oppure

2. $\mathcal{M} =$  $U =$  $V =$ 

$U = M$

Tempo a disposizione: 1h

Le risposte vanno adeguatamente giustificate

(1)
$$g = dx^2 + dy^2$$
$$X = x\frac{\partial}{\partial y} - y^\alpha \frac{\partial}{\partial x}$$

det α in modo che X sia di Killing

X killing:

$\mathcal{L}_X g = 0$

$$\mathcal{L}_{x\frac{\partial}{\partial y}}(dx^2) = \mathcal{L}_{x\frac{\partial}{\partial y}}(dx)\,dx + dx\,\mathcal{L}_{x\frac{\partial}{\partial y}}\,dx$$

$$= d(\mathcal{L}_{x\frac{\partial}{\partial y}} x)\,dx + dx\, d(\mathcal{L}_{x\frac{\partial}{\partial y}} x)$$

$$= d\left(x\frac{\partial x}{\partial y}\right)dx + \dots = 0$$

$$\mathcal{L}_{x\frac{\partial}{\partial y}}(dy)^2 = \mathcal{L}_{x\frac{\partial}{\partial y}}(dy)\,dy + dy\,\mathcal{L}_{x\frac{\partial}{\partial y}}\,dy$$

$$= d(\mathcal{L}_{x\frac{\partial}{\partial y}} y)\,dy + dy\, d(\mathcal{L}_{x\frac{\partial}{\partial y}} y)$$

$$= d\left(x\frac{\partial y}{\partial y}\right)dy + dy\,dx \qquad = \boxed{2\,dx\,dy}$$

↖ simm !

$$= dx \otimes dy + dy \otimes dx$$

$$\mathcal{L}_{y^\alpha \frac{\partial}{\partial x}}(dy^2) = \dots 0$$

$$\mathcal{L}_{y^\alpha \frac{\partial}{\partial x}}(dx)^2 = \mathcal{L}_{y^\alpha\frac{\partial}{\partial x}}(dx)\,dx + dx\,\mathcal{L}_{y^\alpha\frac{\partial}{\partial x}}(dx)$$

$$= d\,\mathcal{L}_{y^\alpha\frac{\partial}{\partial x}}(x)\,dx + dx\,d\left(y^\alpha\frac{\partial x}{\partial x}\right)$$

$$= \boxed{2\,dx\,d(y^\alpha)} \qquad \Rightarrow \alpha = 1$$

(chiaro anche a priori!)

(2) In $M = \mathbb{R}^2 \setminus \{(0,0)\}$ si consideri

$$\omega = (y^2+1)\,dy + 2\,\frac{x\,dy - y\,dx}{x^2+y^2}$$

Si stabilisca se $\omega \in Z^1(M)$, $\omega \in B^1(M)$

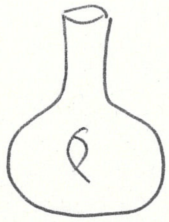
e si stabilisca se è coomologa alla "forma angolare".

$$\omega = d\left(\frac{y^3}{3} + y\right) + 2\,\omega_{ang}$$

$\Rightarrow$ ω è chiusa ma $[\omega] \neq 0$

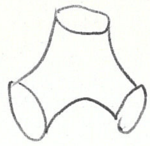
Di più, si ha $[\omega] = 2[\omega_{ang}]$

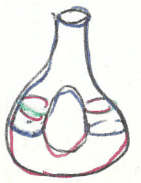
$(\omega \sim 2\,\omega_{ang}\,,\ \omega \not\sim \omega_{ang})$

③ $M =$  calcolare $H^*(M)$

con M.V.

$H^0(V)=\mathbb{R}$
$H^1(V)=\mathbb{R}$
$H^2(V)=0$

1° $\quad M = U \cup V \qquad U:$  $\quad V =$ 



$H^0(U)=\mathbb{R}$ conn.
$H^1(U)=\mathbb{R}^2$
(piano meno due punti)
$H^2(U)=0$
(non compatto)

$U\cap V =$ 

$H^0(U\cap V)=\mathbb{R}^2$
$H^1(U\cap V)=\mathbb{R}^2$
$H^2(U\cap V)=0$

$$\begin{array}{ccccc} \mathbb{R} & & \mathbb{R} \qquad \mathbb{R} & & \mathbb{R}^2 \\ H^0(M) & \xrightarrow{f_1} & H^0(U)\oplus H^0(V) & \xrightarrow{f_2} & H^0(U\cap V) \end{array}$$

$$\xrightarrow{\delta_1}$$

$$\begin{array}{ccccc} & & \mathbb{R}^2 \qquad \mathbb{R} & & \mathbb{R}^2 \\ \boxed{H^1(M)} & \xrightarrow{f_3} & H^1(U)\oplus H^1(V) & \xrightarrow{f_4} & H^1(U\cap V) \end{array}$$

$$\xrightarrow{\delta_2}$$

$$\begin{array}{ccccc} H^2(M) & \xrightarrow{f_5} & H^2(U)\oplus H^2(V) & \longrightarrow & H^2(U\cap V) \\ 0 & & 0 \qquad 0 & & 0 \end{array}$$

$0 = \mathrm{Im}\,\delta_2 \Rightarrow \ker \delta_2 = \mathbb{R}^2 \Rightarrow \mathrm{Im} f_4 = \ker \delta_2 = \mathbb{R}^2$

$\Rightarrow (N+R) \quad \ker f_4 = \mathbb{R} \quad (3-2=1)$

$\mathrm{Im} f_3 = \ker f_4 = \mathbb{R} \Rightarrow h^1(M) = 1 + \dim \ker f_3$

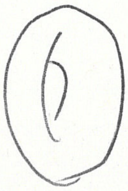
$\boxed{\ker f_3} = \mathrm{Im}\,\delta_1 . \qquad \ker \delta_1 = \mathrm{Im} f_2 = \mathbb{R} \qquad \dim \ker \delta_1 = \dim(\mathrm{Im} f_2) = 1$

$\dim(\mathrm{Im}\,\delta_1) = 2-1 = 1$

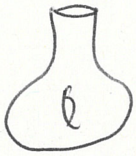
$\boxed{h^1(M) = 1+1 = 2}$

E poi $H^0(M)=\mathbb{R}$ (chiuso)
$H^2(M)=0$ $\quad$ M or, non compatto)

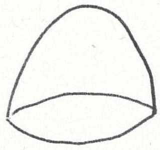
2°



M



U



V

$U \cap V$

$H^0(M) = \mathbb{R}$
$H^1(M) = \mathbb{R}^2$
$H^2(M) = \mathbb{R}$

$H^0(U) = \mathbb{R}$
$H^1(U) = ?$
$H^2(U) = \mathbb{R}$

$H^0(V) = \mathbb{R}$
$H^1(V) = 0$
$H^2(V) = 0$

$H^0(U\cap V) = \mathbb{R}$
$H^1(U\cap V) = \mathbb{R}$
$H^2(U\cap V) = 0$

$$\underbrace{H^0(U) \oplus H^0(V)}_{} \xrightarrow{f_1} \underbrace{H^0(U\cap V)}_{} \quad \left(\mathbb{R}^2 \to \mathbb{R}\right)$$

$$\xrightarrow{\delta_1} \overset{\mathbb{R}^2}{H^1(M)} \xrightarrow{f_2} \boxed{\overset{?}{H^1(U)}} \oplus \overset{0}{H^1(V)} \xrightarrow{f_3} \overset{\mathbb{R}}{H^1(U\cap V)}$$

$$\xrightarrow{\delta_2} \underset{\mathbb{R}}{H^2(M)} \xrightarrow{f_3} H^2(U) \oplus \overset{0}{H^2(V)} \longrightarrow \overset{0}{H^2(U\cup V)}$$

$\mathrm{Im}\, f_3 = 0$

$\Rightarrow \ker f_3 = \mathbb{R}$ $\qquad \mathrm{Im}\, \delta_2 = \ker f_3 = \mathbb{R}$

$\Rightarrow \ker \delta_2 = 0$ $\qquad \mathrm{Im}\, f_3 = \ker \delta_2 = 0$

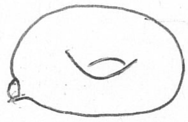
$\Rightarrow$ $H^1(U) = \ker f_3 = \mathrm{Im}\, f_2$

$\dim(\mathrm{Im}\, f_2) = 2 - \dim(\ker f_2) = 2 - \dim(\mathrm{Im}\, \delta_1)$

$= 2 - (1 - \dim(\ker \delta_1)) = 2 - (1-1) = 2$

$\Rightarrow H^1(U) \cong \mathbb{R}^2$

3° metodo

$$M \approx U =$$ 

toro $\setminus \{pto\}$

$\bigstar$ Si ricordi che $\quad \chi(U) = \underset{\substack{\parallel \\ \text{toro}}}{\chi(M)} - 1 \qquad$ (in generale)

Ma $\quad \underset{\text{toro}}{\chi(M)} = 0 \quad \Rightarrow \quad \chi(U) = -1$

$$\text{ma} \quad \underset{U \text{ connesso}}{h^0(U) = 1}, \quad \underset{U \text{ non compatto}}{h^2(U) = 0}$$

$$\chi(U) = -1 = 1 - h^1(U) + 0$$

$$\Rightarrow h^1(U) = 2 \qquad H^1(M) \cong \mathbb{R}^2 \quad \checkmark$$