TOPOLOGIA E GEOMETRIA DIFFERENZIALE

a.a. 2014/15

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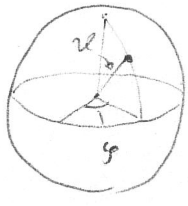
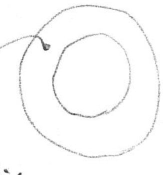
(1) In $\mathbb{R}^3$, sia $\omega = dx \wedge dy \wedge dz$

Sia $f : \mathbb{R}^3 \circlearrowleft$ $\begin{cases} x = \rho \sin\vartheta \cos\varphi \\ y = \rho \sin\vartheta \sin\varphi \\ z = \rho \cos\vartheta \end{cases}$

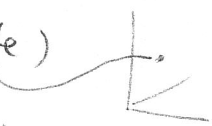


Calcolare $f^*\omega$. Sia $X = \rho^\alpha \frac{\partial}{\partial \rho}$ Determinare $\alpha \in \mathbb{R}$ in modo che X conservi il volume: $\mathcal{L}_X(f^*\omega) = 0$

(2) Sia $M =$  Calcolare $H^*(M)$

guscio sferico (esclusa la frontiera)

(3) In $\{x > 0, y > 0, z > 0\}$ (primo ottante)



Sia data $\omega = xy\,dz + yz\,dx + zx\,dy$

Dire se $\omega = 0$ dà vita ad una distribuzione integrabile e, in caso affermativo, individuarne una base di campi vettoriali commutanti.

Tempo a disposizione: 1h

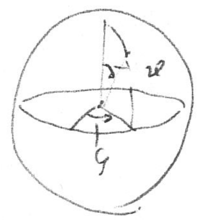
Le risposte vanno adeguatamente giustificate

(1) $M = \mathbb{R}^3$ $\qquad \omega = dx \wedge dy \wedge dz$

$\omega \in Z^3$, $\omega \in B^3$? $\qquad \omega = d(x\, dy \wedge dz)$

sì. $\qquad$ (Poincaré...)

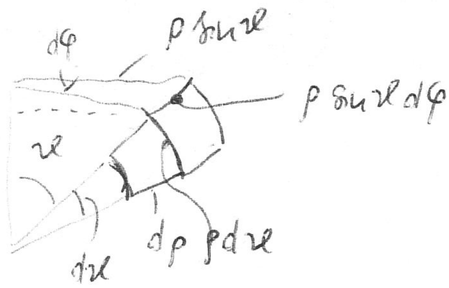
$$f : \mathbb{R}^3 \circlearrowleft \qquad \begin{cases} x = \rho \sin\vartheta \cos\varphi \\ y = \rho \sin\vartheta \sin\varphi \\ z = \rho \cos\vartheta \end{cases}$$



calcolare $f^*\omega$

(si trova subito $\qquad f^*\omega = \rho^2 \sin\vartheta \, d\rho \wedge d\vartheta \wedge d\varphi$)

Sia $X = \rho^\alpha \dfrac{\partial}{\partial \rho}$



det α in modo che X conservi il volume : $\quad \mathcal{L}_X (f^*\omega) = 0$

$$\mathcal{L}_{\rho^\alpha \frac{\partial}{\partial\rho}} (\rho^2 \sin\vartheta \, d\rho \wedge d\vartheta \wedge d\varphi) =$$

$$\mathcal{L}_{\rho^\alpha \frac{\partial}{\partial\rho}} (\rho^2 \sin\vartheta)\, d\rho \wedge d\vartheta \wedge d\varphi + \rho^2 \sin\vartheta \, \mathcal{L}_{\rho^\alpha \frac{\partial}{\partial\rho}} d\rho \wedge d\vartheta \wedge d\varphi$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\underbrace{\rho^\alpha \frac{\partial}{\partial\rho} (\rho^2 \sin\vartheta)}\, d\rho \wedge d\vartheta \wedge d\varphi \qquad \rho^2 \sin\vartheta \, \mathcal{L}_{\rho^\alpha \frac{\partial}{\partial\rho}} d\rho \wedge d\vartheta \wedge d\varphi$$

$$+ \rho^2 \sin\vartheta \, d\rho \wedge \mathcal{L}_{\rho^\alpha \frac{\partial}{\partial\rho}} d\vartheta \wedge d\varphi$$

$$2\rho^{\alpha+1} \sin\vartheta \, d\rho \wedge d\vartheta \wedge d\varphi \qquad + \rho^2 \sin\vartheta \, d\rho \wedge d\vartheta \wedge \mathcal{L}_{\rho^\alpha \frac{\partial}{\partial\rho}} d\varphi$$

$$= 2\rho^{\alpha+1} \sin\vartheta \, d\rho \wedge d\vartheta \wedge d\varphi + \rho^2 \sin\vartheta \, d\rho^\alpha \wedge d\vartheta \wedge d\varphi + 0 + 0$$

$$= 2\rho^{\alpha+1} \sin\vartheta \, d\rho \wedge d\vartheta \wedge d\varphi + \rho^2 \alpha \rho^{\alpha-1} \sin\vartheta \, d\rho \wedge d\vartheta \wedge d\varphi$$

$$= \rho^{\alpha+1}(2+\alpha) \cdot \sin\vartheta \, d\rho\wedge d\vartheta\wedge d\varphi$$

$$\mathcal{L}_X(\ \ ) = 0 \quad \Longrightarrow \quad \alpha = -2$$

(controllo: $\xi = \rho^3$ $\qquad d\xi = 3\rho^2 d\rho$ $\qquad \frac{\partial f}{\partial\rho} = \frac{\partial f}{\partial \xi}\frac{\partial\xi}{\partial\rho}$

$$X = \frac{1}{\rho^2}\frac{\partial}{\partial\rho} = \frac{1}{\rho^2}\cdot 3\cancel{\rho^2}\cdot\frac{\partial}{\partial\xi} = 3\frac{\partial}{\partial\xi} \qquad \frac{\partial}{\partial\rho} = 3\rho^2\frac{\partial}{\partial\xi}$$

$$\rho^2 \sin\vartheta \, d\rho\wedge d\vartheta\wedge d\varphi = \frac{1}{3}\overbrace{\sin\vartheta \, d\xi\wedge d\vartheta\wedge d\varphi}$$

e $\mathcal{L}_X(\cdot) = 0$ come si vede immediatamente)

(2) Determinare $H^*(M)$ $\qquad M =$ guscio sferico

È $M \simeq S^2\times\mathbb{R}$

$H^*(M) = H^*(S^2)$ s' $\begin{cases} H^0 = \mathbb{R} \\ H^1 = 0 \\ H^2 = \mathbb{R} \\ H^3 = 0 \end{cases}$

③ in $\{x>0, y>0, z>0\}$

$$\omega = xy\,dz + yz\,dx + zx\,dy$$

Dire se $\omega = 0$ dà vita ad una distr. integrabile e, in caso affermativo, individuarne una base di vettori commutanti

La soluzione è immediata : $\omega = d(xyz)$ $(\clubsuit)$

$\Rightarrow$ varietà integrali: $xyz = c > 0$

Se non si vedesse subito $\clubsuit$, si calcola $d\omega = \ldots = 0$

$\Rightarrow$ $\omega \wedge d\omega = 0$

Sia $X = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + \gamma \frac{\partial}{\partial z}$ $\qquad \omega(X) = 0$

$$\omega(X) \overset{\shortparallel}{=} df(X) \overset{\shortparallel}{=} X(f)$$

$$X(f) = \alpha \frac{\partial}{\partial x}(xyz) + \beta \frac{\partial}{\partial y}(xyz) + \gamma \frac{\partial}{\partial z}(xyz)$$

$$= \alpha\, yz + \beta\, xz + \gamma\, xy = 0$$

$\Rightarrow$ prendiamo ad es. $\alpha = x \quad \beta = -y \quad \gamma = 0$

$\alpha = 0 \quad \beta = y \quad \gamma = -z$

$$X_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

$$X_2 = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$$

È subito $[X_1, X_2] = 0$

controllo:

$x\frac{\partial}{\partial x}(xyz) - y\frac{\partial}{\partial y}(xyz)$

$= xyz - xyz = 0$ ecc.