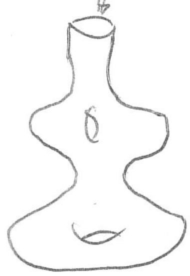
TOPOLOGIA E GEOMETRIA DIFFERENZIALE

a.a. 2015/16

Prova scritta del 1° luglio 2016

(1) Data

$M =$ 

calcolarne $H^*(M)$ in due modi

(2) In $\mathbb{R}^3$, determinare $\alpha = \alpha(x,y)$ in modo che $\Delta := \langle X, Y \rangle$ sia una distribuzione integrabile, dove

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$

$$Y = \alpha\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

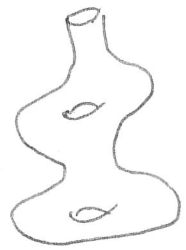
[$\alpha = -\dfrac{2y^3}{x^2}$ è una sol (purché $x \neq 0, y \neq 0$)]

con riferimento a (2)

(3) In $\mathbb{R}^3$, determinare $\omega \in \Lambda^1$ tale che $\Delta = \ker \omega$, e si verifichi a ritroso che $\omega \wedge d\omega = 0$.

Tempo a disposizione 1h.

Le risposte vanno adeguatamente giustificate

① $M =$ 

Calcolare $H^*(M)$

$H^0 = \mathbb{R}$

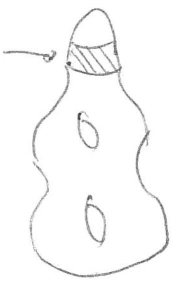
$H^1 = ?$ $\qquad H^1 \cong \mathbb{R}^4$

$H^2 = 0$

1° modo : $\qquad M = U \qquad V =$ 

$U \cup V \equiv \mathcal{M} =$ 

$$\chi(U) = \underset{\substack{\| \\ -2}}{\chi(\mathcal{M})} - 1$$

$U \cap V =$ 

$$\chi(U) = -3$$

$$= \underset{\substack{\| \\ 1}}{h^0} - h^1 + \underset{\substack{\| \\ 0}}{h^2}$$

$$h^1 = 3 + 1 = 4$$

2° modo $\quad M - V$

$$\overset{\mathbb{R}}{H^0(U \cup V)} \xrightarrow{f_1} \overset{\mathbb{R}}{H^0(U)} \oplus \overset{\mathbb{R}}{H^0(V)} \xrightarrow{f_2} \overset{\mathbb{R}}{H^0(U \cap V)} \xrightarrow{\delta_1}$$

$$\xrightarrow{\delta_1} \overset{\mathbb{R}^4}{H^1(U \cup V)} \xrightarrow{f_3} \boxed{H^1(U)} \oplus \overset{0}{H^1(V)} \xrightarrow{f_4} \overset{\mathbb{R}}{H^1(U \cap V)} \xrightarrow{\delta_2}$$

$$\xrightarrow{\delta_2} \overset{\mathbb{R}}{H^2(U \cup V)} \xrightarrow{f_5} \underset{0}{H^2(U)} \oplus \underset{0}{H^2(V)} \xrightarrow{f_6} \underset{0}{H^2(U \cap V)}$$

$\longrightarrow$

$\mathrm{Im}\, f_4 = \mathrm{Ker}\, \delta_2$

$\mathrm{Im}\, \delta_2 = \mathrm{Ker}\, f_5 = \mathbb{R}$

$\Rightarrow \mathrm{Ker}\, \delta_2 = 0 = \mathrm{Im}\, f_4 \longrightarrow \boxed{\mathrm{Im}\, f_4 = 0}$

$\boxed{H^1} = \mathrm{Ker}\, f_4 = \mathrm{Im}\, f_3 \qquad \boxed{\mathrm{Im}\, f_3 = \mathbb{R}^4}$

$\mathrm{Ker}\, f_3 = \mathrm{Im}\, \delta_1 \qquad \boxed{\mathrm{Im}\, \delta_1 = 0}$

$\mathrm{Ker}\, \delta_1 = \mathrm{Im}\, f_2 = \mathbb{R}$

$\mathrm{Ker}\, f_2 = \mathbb{R}$

$H^1 \cong \mathbb{R}^4$

② Determinare $\alpha = \alpha(x, y)$ in modo che (in $\mathbb{R}^3$)

$\Delta = \langle X, Y \rangle$ sia una distribuzione integrabile

$$X = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}$$

$$Y = \alpha(x,y) \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}$$

$$[X, Y] = y \alpha_x \frac{\partial}{\partial y} - \alpha \frac{\partial}{\partial x} \overset{?}{=} \text{c.l.}(X, Y)$$

$$\begin{array}{r} X \to \\ Y \to \\ [X,Y] \to \end{array} \begin{vmatrix} y & 0 & x \\ 0 & \alpha & y \\ -\alpha & y\alpha_x & 0 \end{vmatrix} = 0$$

$$\alpha^2 x - y^3 \alpha_x = 0$$

$$\frac{\alpha_x}{\alpha^2} = \frac{x}{y^3}$$

$$B := \alpha^{-1}$$

$$-\frac{\partial}{\partial x}(\alpha^{-1}) = \frac{x}{y^3} \qquad \beta_x = -\frac{x}{y^3}$$

$$\beta = -\frac{x^2}{2} \cdot \frac{1}{y^3} + \varphi(y)$$

(3)

Determinare w tale che $w = 0$ riproduca Δ

$$\begin{array}{l} \\ \\ \text{rotore} \rightsquigarrow \end{array} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ y & 0 & x \\ 0 & -\frac{2y^2}{x^2} & 1 \end{vmatrix} = \underline{i}\,\frac{2y^2}{x} - \underline{j}\,y - \underline{k}\,\frac{2y^3}{x^2}$$

$$w := \frac{2y}{x}\,dx - dy - \frac{2y^2}{x^2}\,dz = 0$$

$$dw = \frac{2}{x}\,dy \wedge dx - \frac{4y}{x^2}\,dy \wedge dz + 2y^2 \cdot 2\,\frac{1}{x^3}\,dx \wedge dz$$

$$= \frac{2}{x}\,dy \wedge dx - \frac{4y}{x^2}\,dy \wedge dz + 4\,\frac{y^2}{x^3}\,dx \wedge dz$$

$$w \wedge dw = +\frac{4y^2}{x^3}\,dx \wedge dy \wedge dz - \frac{8y^2}{x^3}\,dx \wedge dy \wedge dz + \frac{4y^2}{x^3}\,dx \wedge dy \wedge dz$$

$$= 0$$